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HP-25

Applications Programs

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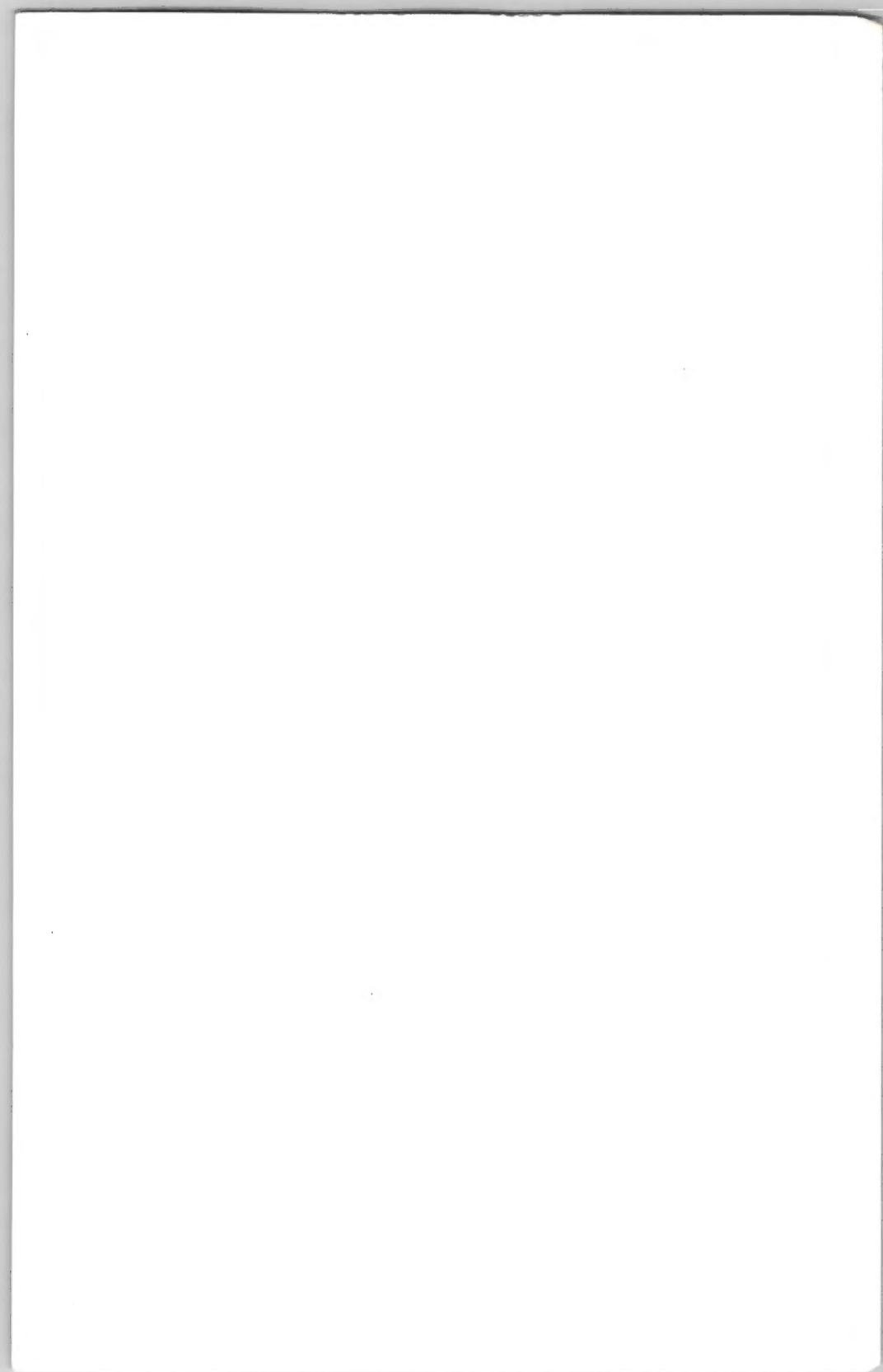
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INTRODUCTION

Welcome. You are about to step into a field that, ten years ago, was open only to users of large computer systems costing tens or hundreds of thousands of dollars, and even five years ago, required a several-thousand-dollar calculator that occupied the better part of a desktop. Today, the HP-25 puts programming into the hands of the individual. It is hoped that this book will allow you to realize some of the potential of this calculating instrument.

These HP-25 Applications Programs have been drawn from the varied fields of mathematics, statistics, finance, surveying, navigation, and games. They have been arranged in eight chapters which follow roughly the above classification. Each program is furnished with a full explanation which includes a description of the problem, any pertinent equations, a list of keystrokes to be entered into program memory, a set of instructions for running the program, and an example or two, with solutions. To use the programs does not require any proficiency in programming, but some familiarity with the HP-25 Owner's Handbook is assumed.

For users who want to enhance their understanding of programming principles and techniques, a number of programs are provided to help in this respect. The first program in each chapter contains, in addition to the usual explanations, a more detailed description of the problem, a commented list of the program keystrokes with a step-by-step tracing of the contents of the stack registers, and a list of the keystrokes required to solve the example problem. Whenever an interesting programming technique is used in one of these programs, it is described in a short section headed "Programming Remarks", which, if present, will immediately precede the list of program keystrokes.

Thus, whether your interest lies in solving a particular problem in a specific area, or in learning more about the programming power of your calculator, we hope that this book will help you get the most from your HP-25.

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A WORD ABOUT PROGRAM USAGE

Various kinds of information are provided to explain the use of each program. Besides a short description of the problem, a list of applicable equations, and an example problem with solution, there are two forms that deserve some explanation: the Program form and the User Instructions form.

Two different Program forms are provided, one of which is just a simplified version of the other. The detailed form is used for a total of eight programs, one per chapter, with the simpler form serving for the rest. A section of a detailed form, taken from the Plotting/Graphing program in Chapter 1, is shown below:

DISPLAY		KEY ENTRY	X	Y	Z	T	COMMENTS	REGISTERS
LINE	CODE							
00			v	θ				$R_0 \Delta t$
01	14 09	$t \rightarrow R$	v_x	v_y			Use polar-to-rectangular for	
02	23 02	STO 2	v_x	v_y			$v_x = v \cos \theta = \text{horiz. vel.}$	
03	21	$x \leftarrow y$	v_y	v_x				$R_1 g$
04	23 03	STO 3	v_y	v_x			$v_y = v \sin \theta = \text{vert. vel.}$	
05	00	0	0					
06	23 04	STO 4	0				Initialize: $t = 0$	$R_2 v_x$
07	24 00	RCL 0	Δt				Start of loop	
08	23 51 04	STO + 4	Δt				Next time interval:	
09	24 04	RCL 4	t				$t \leftarrow t + \Delta t$	$R_3 v_y$
10	15 02	$g x^2$	t^2					

The rightmost column, headed **REGISTERS**, explains what variables are stored in storage registers R_0 through R_7 . The rest of the form is divided into eight columns. The first two columns describe the appearance of the display as the program is being keyed in: **LINE** shows the step number for the current instruction and **CODE** denotes the numeric keycodes corresponding to the keystrokes in the next column, **KEY ENTRY**. The entries in this column are the keys that must be pressed to enter the program into program memory. The **ENTER** key is denoted in this column as $\boxed{\downarrow}$; all other key designations are identical to those appearing on the HP-25.

The next four columns, **X**, **Y**, **Z**, and **T**, trace the contents of the stack registers as they would change during execution of the program in **RUN** mode. Each entry under **X**, **Y**, **Z**, or **T** gives the contents of the respective register *after* the instruction on that line has been executed. The **COMMENTS** column contains additional step-by-step explanation of the program's calculations.

These last columns, **X**, **Y**, **Z**, **T**, and **COMMENTS**, are provided to help the interested user acquire a detailed, in-depth understanding of a particular program, or of programming techniques in general.

The simplified Program forms contain the same information as the detailed forms except for the omission of columns **X**, **Y**, **Z**, **T**, and **COMMENTS**.

The User Instructions form is the user's guide to operating the program to solve his own particular problem. This form, which is composed of five columns, is illustrated below for the same program from Chapter 1, Plotting/Graphing.

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Store time interval	Δt	STO	0			
3	Store gravitational constant	g	STO	1			
4	Input angle and initial speed	θ	\uparrow				
		v	f	PRGM			
5	Perform steps 5 and 6 any number of times: Display time and horizontal distance						
			R/S				(t)
							x
6	Display height		R/S				y
7	To change θ or v , go to step 4.						
	To change Δt or g , go to appropriate step, store new value, then go to step 4.						

Reading from left to right, the **STEP** column gives the instruction step number. The **INSTRUCTIONS** column gives instructions and comments concerning the operations to be performed. Steps are executed in sequential order except where the **INSTRUCTIONS** column directs otherwise.

Normally, the first instruction is "Key in program", which means to store the keystrokes of the program in program memory (switch to **PRGM** mode, press **f** **PRGM**, key in the program, then switch back to **RUN** mode).

Repeated processes, used in most cases for a long string of input/output data, are outlined with a bold border, as in steps 5 and 6 above. In this case, the steps are repeated in order to generate a number of (x,y) pairs for a graph.

The **INPUT DATA/UNITS** column specifies the input data to be supplied, and the units of data if applicable. The **KEYS** column specifies the keys to be pressed. **\uparrow** is used for the **ENTER** key, and all other key designations are identical to those appearing on the HP-25. Ignore any blank positions in the **KEYS** column.

Some programs are complex enough that users have to press additional keys to generate some results. Those keys are also shown in the **KEYS** column.

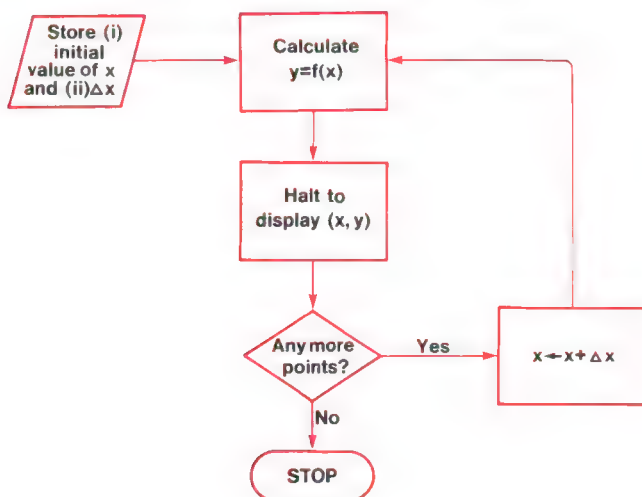
The **OUTPUT DATA/UNITS** column shows intermediate and final results that have been calculated either from the keyboard or from an executing program, and the units of data if applicable. Parentheses around an output variable, such as (t) in step 5, indicate that the result is displayed only briefly by a **PAUSE** instruction (**f** **PAUSE**).

CHAPTER 1 ALGEBRA AND NUMBER THEORY

PLOTTING/GRAPHING

Most people who have labored through a ninth-grade algebra course probably still respond with a shudder to the word "graph". Evidently the tedium of finding $y = 3x^2 - 4x + 4$, for integer values of x from $-\infty$ to $+\infty$, has etched permanent memories in us all. Fortunately, we need not endure this tedium any longer. The HP-25 lends itself perfectly to this kind of repetitive calculation.

The basic idea is to generate (x, y) pairs by keying into program memory the keystrokes required to calculate y , assuming x is given. Then the user need only return to the top of memory, enter a value for x , press **[R/S]**, and see y displayed within seconds. The process may be repeated for as many values of x as desired. The programmer can take this process one step further into automation by also having the calculator generate each new value of x , for example, by adding 1 to the old value, or, in general, by adding a specified increment Δx . A flowchart of the process is shown below.



The program used here to illustrate this process takes a slightly different tack. We will consider the problem of plotting the trajectory of a stone which is hurled into the air with an initial velocity v at an angle to the horizontal of θ . Neglecting drag due to friction with the atmosphere, the following equations describe the stone's x - and y -coordinates as functions of the time t :

$$x = vt \cos \theta$$

$$y = vt \sin \theta - \frac{1}{2} gt^2$$

where x = horizontal distance the stone has traveled

y = height of the stone

g = acceleration due to gravity

$$\approx 9.8 \text{ m/s}^2$$

$$\approx 32 \text{ ft/s}^2$$

These equations differ slightly from the usual graphing function in that y is not expressed directly as a function of x , but instead both x and y are expressed as functions of a third variable t . The points to be plotted are still the ordered pairs (x, y) ; but now it is the time t which should be incremented by an amount Δt .

Notes:

1. Any consistent set of units may be used.
2. This is *not* a general plotting/graphing program; it merely illustrates the method by application to a specific problem. However, some study of the program listing and the flowchart should enable the user to adapt the method to his own application.

Programming Remarks:

1. The components of the velocity in the horizontal and vertical directions, v_x and v_y , are computed in one step by a conversion of v and θ to rectangular coordinates ($\boxed{f} \boxed{\rightarrow R}$). The values $v_x = v \cos \theta$ and $v_y = v \sin \theta$ are returned to the X- and Y-registers, respectively.
2. A pause ($\boxed{f} \boxed{\text{PAUSE}}$) is used in this program in a very typical manner, to display briefly the output variable t , whose values are simple (0.25, 0.50, 0.75, etc.) and do not need to be written down.

DISPLAY		KEY ENTRY	X	Y	Z	T	COMMENTS	REGISTERS
LINE	CODE							
00			v	θ				R 0 Δt
01	14 09	f \rightarrow R	v_x	v_y			Use polar-to-rectangular for	
02	23 02	STO 2	v_x	v_y			$v_x = v \cos \theta = \text{horiz. vel.}$	
03	21	$x \leftrightarrow y$	v_y	v_x				R 1 g
04	23 03	STO 3	v_y	v_x			$v_y = v \sin \theta = \text{vert. vel.}$	
05	00	0	0					
06	23 04	STO 4	0				Initialize: $t = 0$	R 2 v_x
07	24 00	RCL 0	Δt				Start of loop	
08	23 51 04	STO + 4	Δt				Next time interval:	
09	24 04	RCL 4	t				$t \leftarrow t + \Delta t$	R 3 v_y
10	15 02	$g \times^2$	t^2					
11	24 01	RCL 1	g	t^2				R 4 t
12	61	x	$g t^2$					
13	02	2	2	$g t^2$				
14	71	\div	$1/2 g t^2$					
15	32	CHS	$-1/2 g t^2$					R 5
16	24 04	RCL 4	t	$-1/2 g t^2$				
17	24 03	RCL 3	v_y	t	$-1/2 g t^2$			
18	61	x	$v_y t$	$-1/2 g t^2$				R 6
19	51	+	y				$y = v_y t - 1/2 g t^2$	
20	24 04	RCL 4	t	y				R 7
21	24 02	RCL 2	v_x	t	y			
22	61	x	x	y			$x = v_x t$	
23	24 04	RCL 4	t	x	y			
24	14 74	f PAUSE	t	x	y		Pause to display t	
25	22	R \downarrow	x	y		t		
26	74	R/S	x	y		t	Halt and display x	
27	21	$x \leftrightarrow y$	y	x		t		
28	74	R/S	y	x		t	Halt and display v	
29	13 07	GTO 07	y	x		t	Branch back for next t	
30								
31								
32								
33								
34								
35								
36								
37								
38								
39								
40								
41								
42								
43								
44								
45								
46								
47								
48								
49								

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Store time interval	Δt	STO	0			
3	Store gravitational constant	g	STO	1			
4	Input angle and initial speed	θ	\uparrow				
		v	\uparrow	PRGM			
5	Perform steps 5 and 6 any number of times: Display time and horizontal distance		R/S				(t)
							x
6	Display height		R/S				y
7	To change θ or v , go to step 4.						
	To change Δt or g , go to appropriate step, store new value, then go to step 4.						

Example:

Plot the trajectory of a stone cast upwards with a velocity of 20 m/s at an angle of 30° to the horizontal. Use intervals of $\frac{1}{4}$ second between points plotted. Let $g = 9.8 \text{ m/s}^2$.

Solution:

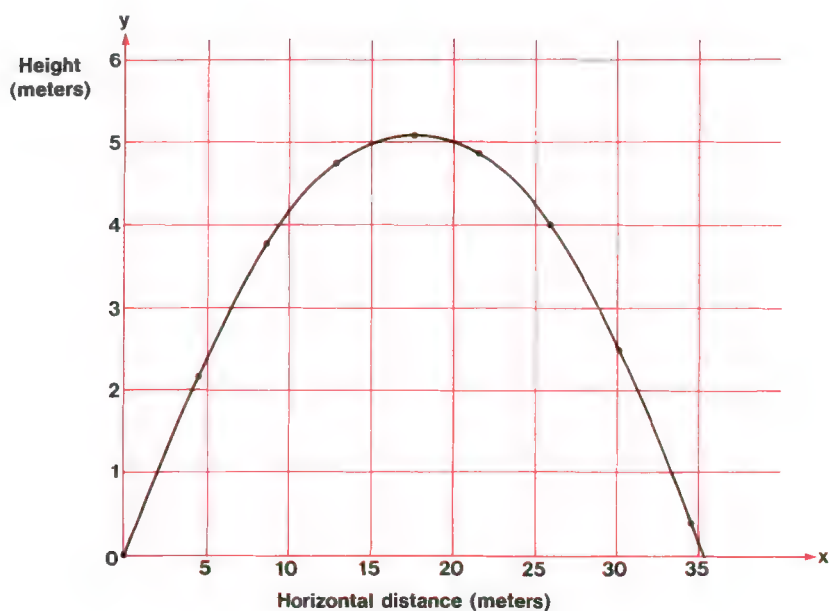
0.25 **STO** **0** 9.8 **STO** **1** 30 \uparrow 20 \uparrow **PRGM** **R/S** \longrightarrow 0.25 (t_1)
 4.33 (x_1)
R/S \longrightarrow 2.19 (y_1)
R/S \longrightarrow 0.5 (t_2)
 8.66 (x_2)
R/S \longrightarrow 3.78 (y_2)
R/S \longrightarrow 0.75 (t_3)
 12.99 (x_3)
R/S \longrightarrow 4.74 (y_3)
 etc.

Continue until y becomes negative.

The table of these results is shown below:

t	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.25
x	4.33	8.66	12.99	17.32	21.65	25.98	30.31	34.64	38.97
y	2.19	3.78	4.74	5.10	4.84	3.98	2.49	0.40	-2.31

The plot of these (x, y) values is made and the stone's trajectory is seen to be a parabola.



QUADRATIC EQUATION

The roots x_1, x_2 of $ax^2 + bx + c = 0$

are given by
$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $D = (b^2 - 4ac)/4a^2$

is positive or zero, the roots are real. In these cases, better accuracy may sometimes be obtained by first computing the root with the larger absolute value:

If $-\frac{b}{2a} \geq 0$, $x_1 = -\frac{b}{2a} + \sqrt{D}$

If $-\frac{b}{2a} < 0$, $x_1 = -\frac{b}{2a} - \sqrt{D}$

In either case, $x_2 = \frac{c}{x_1 a}$.

If $D < 0$, the roots are complex, being

$$u \pm iv = \frac{-b}{2a} \pm \frac{\sqrt{4ac - b^2}}{2a} i$$

DISPLAY		KEY ENTRY
LINE	CODE	
00		
01	31	\uparrow
02	22	$R\downarrow$
03	71	\div
04	02	2
05	71	\div
06	32	CHS
07	31	\uparrow
08	15 02	$g x^2$
09	22	$R\downarrow$
10	22	$R\downarrow$
11	21	$x \rightleftarrows y$
12	71	\div
13	23 00	STO 0
14	41	-
15	14 74	f PAUSE
16	15 41	$g x < 0$
17	13 31	GTO 31
18	14 02	$f \sqrt{x}$
19	21	$x \rightleftarrows y$
20	15 41	$g x < 0$
21	13 24	GTO 24
22	51	+
23	13 26	GTO 26
24	21	$x \rightleftarrows y$

DISPLAY		KEY ENTRY
LINE	CODE	
25	41	-
26	74	R/S
27	15 22	$g 1/x$
28	24 00	RCL 0
29	61	x
30	13 00	GTO 00
31	32	CHS
32	14 02	$f \sqrt{x}$
33	21	$x \rightleftarrows y$
34	74	R/S
35	21	$x \rightleftarrows y$
36	13 00	GTO 00
37		
38		
39		
40		
41		
42		
43		
44		
45		
46		
47		
48		
49		

REGISTERS	
R_0	c/a
R_1	
R_2	
R_3	
R_4	
R_5	
R_6	
R_7	

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Initialize		f	PRGM			
3	Enter coefficients and display D	c	↑				
		b	↑				
		a	R/S				(D)
4	If $D \geq 0$, roots are real						x_1
			R/S				x_2
	or						
	If $D < 0$, roots are complex of						
	form $u \pm iv$						u
			R/S				v
5	For new case, go to step 3.						

Example:

Find solutions to the three equations below:

- $x^2 + x - 6 = 0$
- $3x^2 + 2x - 1 = 0$
- $2x^2 - 3x + 5 = 0$

Solutions:

- $D = 6.25$
 $x_1 = -3.00$
 $x_2 = 2.00$
- $D = 0.44$
 $x_1 = -1.00$
 $x_2 = 0.33$
- $D = -1.94$
 $x_1, x_2 = 0.75 \pm 1.39 i$

COMPLEX ARITHMETIC, +, -, x, ÷

Let $a_1 + ib_1$ and $a_2 + ib_2$ be two complex numbers. The arithmetic operations +, -, x, ÷ are defined as follows:

1. +, addition

$$(a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + (b_1 + b_2)i$$

2. -, subtraction

$$(a_1 + ib_1) - (a_2 + ib_2) = (a_1 - a_2) + (b_1 - b_2)i$$

3. x, multiplication

$$(a_1 + ib_1) \times (a_2 + ib_2) = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

4. ÷, division

$$\frac{(a_1 + ib_1)}{(a_2 + ib_2)} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}, a_2 + ib_2 \neq 0$$

where $r_1 e^{i\theta_1}$ is the polar representation of $a_1 + ib_1$ and $r_2 e^{i\theta_2}$ is the polar representation of $a_2 + ib_2$. In each case let the answer be $x + iy$.

After a calculation is finished x is stored in R_0 as well as the X-register and y is stored in R_1 as well as the Y-register. In this way arithmetic operations can be chained together.

DISPLAY		KEY ENTRY
LINE	CODE	
00		
01	32	CHS
02	21	$x \div y$
03	32	CHS
04	21	$x \div y$
05	24 00	RCL 0
06	51	+
07	21	$x \div y$
08	24 01	RCL 1
09	51	+
10	13 31	GTO 31
11	15 09	$g \rightarrow P$
12	15 22	$g \ 1/x$
13	21	$x \div y$
14	32	CHS
15	21	$x \div y$
16	13 18	GTO 18
17	15 09	$g \rightarrow P$
18	24 02	STO 2
19	22	$R \downarrow$
20	24 01	RCL 1
21	24 00	RCL 0
22	15 09	$g \rightarrow P$
23	24 02	RCL 2
24	61	x

DISPLAY		KEY ENTRY
LINE	CODE	
25	23 02	STO 2
26	22	$R \downarrow$
27	51	+
28	24 02	RCL 2
29	14 09	$f \rightarrow R$
30	21	$x \div y$
31	23 01	STO 1
32	21	$x \div y$
33	23 00	STO 0
34	13 00	GTO 00
35		
36		
37		
38		
39		
40		
41		
42		
43		
44		
45		
46		
47		
48		
49		

REGISTERS
$R_0 a_1, x$
$R_1 b_1, y$
R_2 Used
R_3
R_4
R_5
R_6
R_7

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Store first complex number	b_1	STO	1			
		a_1	STO	0			
3	Key in next number	b_2	\uparrow				
		a_2					
4	For addition		GTO	05	R/S		x
	or						
	subtraction		f	PRGM	R/S		x
	or						
	multiplication		GTO	17	R/S		x
	or						
	division		GTO	11	R/S		x
5	For imaginary part		$x \rightarrow y$				y
6	For next calculation in chain, go						
	to step 3.						
7	For new case, go to step 2.						

Examples:

$$1. \quad (1.2 + 3.7i) - (2.6 - 1.9i) = -1.4 + 5.6i$$

$$2. \quad \frac{3 + 4i}{7 - 2i} = 0.25 + 0.64i$$

$$3. \quad \left[\frac{(3 + 4i) + (7.4 - 5.6i)}{(7 - 2i)} \right] [3.1 + 4.6i] = 3.61 + 7.16i$$

COMPLEX FUNCTIONS $|z|$, z^2 , $1/z$, \sqrt{z}

A complex number $z = a + ib$ has polar representation $re^{i\theta}$. The formulas used to evaluate the given functions are as follows:

1. $|z| = r$
2. $z^2 = r^2 e^{i2\theta}$
3. $1/z = \frac{1}{r} e^{-i\theta}$, $z \neq 0$
4. $\sqrt{z} = \pm(\sqrt{r} e^{i\theta/2}) = \pm(x + iy)$

The answer is represented by $x + iy$.

DISPLAY		KEY ENTRY
LINE	CODE	
00		
01	15 09	$g \rightarrow P$
02	13 00	GTO 00
03	15 09	$g \rightarrow P$
04	15 02	$g x^2$
05	21	$x \leftrightarrow y$
06	31	\uparrow
07	51	$+$
08	21	$x \leftrightarrow y$
09	14 09	$f \rightarrow R$
10	13 00	GTO 00
11	15 09	$g \rightarrow P$
12	15 22	$g 1/x$
13	21	$x \leftrightarrow y$
14	32	CHS
15	21	$x \leftrightarrow y$
16	14 09	$f \rightarrow R$
17	13 00	GTO 00
18	15 09	$g \rightarrow P$
19	14 02	$f \sqrt{x}$
20	21	$x \leftrightarrow y$
21	02	2
22	71	\div
23	21	$x \leftrightarrow y$
24	14 09	$f \rightarrow R$

DISPLAY		KEY ENTRY
LINE	CODE	
25	13 00	GTO 00
26		
27		
28		
29		
30		
31		
32		
33		
34		
35		
36		
37		
38		
39		
40		
41		
42		
43		
44		
45		
46		
47		
48		
49		

REGISTERS	
R ₀	
R ₁	
R ₂	
R ₃	
R ₄	
R ₅	
R ₆	
R ₇	

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Key in z	b	↑				
		a					
3	For z		f	PRGM	R/S		z
	or						
	z^2		GTO	03	R/S		x
			$x \rightarrow y$				y
	or						
	$1/z$		GTO	11	R/S		x
			$x \rightarrow y$				y
	or						
	\sqrt{z}		GTO	18	R/S		x
			$x \rightarrow y$				y
4	For new case, go to step 2.						

Examples:

- $|12 - 5i| = 13.00$
- $(6 - i)^2 = 35.00 - 12.00i$
- $\frac{1}{2 + 5i} = 0.07 - 0.17i$
- $\sqrt{3 + 4i} = \pm (2.00 + 1.00i)$

DETERMINANT AND INVERSE OF A 2 x 2 MATRIX

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ be a 2 x 2 matrix.

The determinant of A denoted by Det A or |A| is evaluated by the following formula:

$$\text{Det } A = a_{22} a_{11} - a_{12} a_{21}$$

Also, the program evaluates the multiplicative inverse A^{-1} of A. The following formula is used:

$$A^{-1} = \begin{bmatrix} a_{22}/\text{Det } A & -a_{12}/\text{Det } A \\ -a_{21}/\text{Det } A & a_{11}/\text{Det } A \end{bmatrix}$$

DISPLAY		KEY ENTRY
LINE	CODE	
00		
01	24 04	RCL 4
02	24 01	RCL 1
03	61	x
04	24 02	RCL 2
05	24 03	RCL 3
06	61	x
07	41	-
08	23 00	STO 0
09	74	R/S
10	24 04	RCL 4
11	24 00	RCL 0
12	71	÷
13	74	R/S
14	24 02	RCL 2
15	24 00	RCL 0
16	71	÷
17	32	CHS
18	74	R/S
19	24 03	RCL 3
20	24 00	RCL 0
21	71	÷
22	32	CHS
23	74	R/S
24	24 01	RCL 1

DISPLAY		KEY ENTRY
LINE	CODE	
25	24 00	RCL 0
26	71	÷
27	13 00	GTO 00
28		
29		
30		
31		
32		
33		
34		
35		
36		
37		
38		
39		
40		
41		
42		
43		
44		
45		
46		
47		
48		
49		

REGISTERS
R ₀ Det A
R ₁ a ₁₁
R ₂ a ₁₂
R ₃ a ₂₁
R ₄ a ₂₂
R ₅
R ₆
R ₇

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Store matrix	a_{11}	STO	1			
		a_{12}	STO	2			
		a_{21}	STO	3			
		a_{22}	STO	4			
3	Compute determinant		f	PRGM	R/S		Det A
4	Compute inverse		R/S				a_{11}^{-1}
			R/S				a_{12}^{-1}
			R/S				a_{21}^{-1}
			R/S				a_{22}^{-1}
5	For new case, go to step 2.						

Example:

Find the determinant and inverse of the matrix

$$A = \begin{bmatrix} 3 & 2 \\ 4 & -4 \end{bmatrix}$$

Solution:

$$\text{Det } A = -20$$

$$A^{-1} = \begin{bmatrix} 0.20 & 0.10 \\ 0.20 & -0.15 \end{bmatrix}$$

NUMBER IN BASE b TO NUMBER IN BASE 10

This program consists of two subprograms. The first changes the integer part of a number in base b to a number in base 10.

$$I_{10} = i_n i_{n-1} \dots i_2 i_1 = i_n b^{n-1} + i_{n-1} b^{n-2} + \dots + i_2 b + i_1$$

This is evaluated in the form

$$b (\dots (b (b (i_n b + i_{n-1}) + i_{n-2}) + \dots) + i_2) + i_1$$

The second subprogram changes the fraction part of a number in base b to a number in base 10.

$$F_{10} = f_1 f_2 \dots f_m = f_1 b^{-1} + f_2 b^{-2} + \dots + f_m b^{-m}$$

Together the two programs can convert any number in base b to a number in base 10. Zeros must be entered in their proper place.

DISPLAY			KEY ENTRY			REGISTERS		
LINE	CODE		LINE	CODE				
00			25	61	x	R ₀	b	
01	23 01	STO 1	26	51	+	R ₁	Used	
02	24 00	RCL 0	27	13 20	GTO 20	R ₂	b ⁻¹	
03	31	↑	28			R ₃	b ⁻ⁱ	
04	31	↑	29			R ₄		
05	31	↑	30			R ₅		
06	24 01	RCL 1	31			R ₆		
07	74	R/S	32			R ₇		
08	23 01	STO 1	33					
09	34	CLX	34					
10	51	+	35					
11	61	x	36					
12	24 01	RCL 1	37					
13	51	+	38					
14	13 07	GTO 07	39					
15	24 00	RCL 0	40					
16	15 22	g 1/x	41					
17	23 02	STO 2	42					
18	23 03	STO 3	43					
19	61	x	44					
20	74	R/S	45					
21	24 02	RCL 2	46					
22	24 03	RCL 3	47					
23	61	x	48					
24	23 03	STO 3	49					

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Store base	b	STO	0			
3	For integer part, input left most digit	i_n	f	PRGM	R/S		
4	Perform for $j = n-1, \dots, 2$ Input next digit	i_j^*	R/S				
5	Input final digit	i_1^*	R/S				I_{10}
6	For fractional part, input digit after decimal	f_1	GTO	15	R/S		
7	Perform for $j = 2, \dots, m-1$ Input next digit	f_j^*	R/S				
8	Input final digit	f_m^*	R/S				F_{10}
9	For new case, go to step 2.						
	* The stack must be maintained at these points.						

Examples:

- $1777_8 = 1023_{10}$
- $143.2044_5 = 48.4384_{10}$

NUMBER IN BASE 10 TO NUMBER IN BASE b

This program will convert any positive number in base 10, N_{10} , to a number in base b , N_b , where $2 \leq b \leq 100$. The algorithm used is an iterative one which adds one more digit to N_b at each iteration. The program pauses as each new N_b is computed to display successive approximations to the final answer. When the displayed value of N_b has reached the accuracy desired by the user, he should press **R/S** to halt the program, then **RCL** **3** to display N_b .

Notes:

- When the base b is such that $11 \leq b \leq 100$, two display positions are allocated to each digit of N_b . Begin partitioning to the right and to the left of the decimal point. For example, 41106.12 in base 16 stands for 4B6.C.
- An error indication during execution means that the machine's accuracy has been exceeded. The value of N_b is in R_3 .

DISPLAY		KEY ENTRY
LINE	CODE	
00		
01	24 00	RCL 0
02	01	1
03	00	0
04	14 51	f $x \geq y$
05	13 09	GTO 09
06	01	1
07	00	0
08	00	0
09	23 02	STO 2
10	00	0
11	23 03	STO 3
12	24 01	RCL 1
13	14 07	f LN
14	24 00	RCL 0
15	14 07	f LN
16	71	\div
17	15 41	g $x < 0$
18	13 21	GTO 21
19	14 01	f INT
20	13 24	GTO 24
21	14 01	f INT
22	01	1
23	41	-
24	23 04	STO 4

DISPLAY		KEY ENTRY
LINE	CODE	
25	24 02	RCL 2
26	21	$x \div y$
27	14 03	f y^x
28	24 03	RCL 3
29	51	+
30	23 03	STO 3
31	14 74	f PAUSE
32	14 74	f PAUSE
33	24 00	RCL 0
34	24 04	RCL 4
35	14 03	f y^x
36	23 41 01	STO - 1
37	13 12	GTO 12
38		
39		
40		
41		
42		
43		
44		
45		
46		
47		
48		
49		

REGISTERS
R_0 b
R_1 N_{10}
R_2 10 or 100
R_3 N_b
R_4 1 digit
R_5
R_6
R_7

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Set display format		f	FIX	9		
3	Store base and decimal number	b	STO	0			
		N_{10}	STO	1	f	PRGM	
4	Display successive approximat-						
	ions to N_b		R/S				(N_b)
5	When number is shown with						
	desired accuracy, press $\boxed{R/S}$ to						
	halt, then		RCL	3			N_b
6	For new case, go to step 3.						

Examples:

- $67.32_{10} = 403.050114_{16}$
 $\quad\quad\quad = 43.51E_{16}$
- $\pi = 3.141592654_{10} = 11.00100100_2$

VECTOR CROSS PRODUCT

If $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ are two three dimensional vectors then the cross product of A and B is denoted by $A \times B$ and is calculated as follows:

$$A \times B = \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right) = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1)$$

Let the solution be represented by (c_1, c_2, c_3) .

DISPLAY		KEY ENTRY
LINE	CODE	
00		
01	24 02	RCL 2
02	24 06	RCL 6
03	61	x
04	24 03	RCL 3
05	24 05	RCL 5
06	61	x
07	41	-
08	74	R/S
09	24 03	RCL 3
10	24 04	RCL 4
11	61	x
12	24 01	RCL 1
13	24 06	RCL 6
14	61	x
15	41	-
16	74	R/S
17	24 01	RCL 1
18	24 05	RCL 5
19	61	x
20	24 02	RCL 2
21	24 04	RCL 4
22	61	x
23	41	-
24	13 00	GTO 00

DISPLAY		KEY ENTRY
LINE	CODE	
25		
26		
27		
28		
29		
30		
31		
32		
33		
34		
35		
36		
37		
38		
39		
40		
41		
42		
43		
44		
45		
46		
47		
48		
49		

REGISTERS	
R ₀	
R ₁ a ₁	
R ₂ a ₂	
R ₃ a ₃	
R ₄ b ₁	
R ₅ b ₂	
R ₆ b ₃	
R ₇	

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Store A	a_1	STO	1			
		a_2	STO	2			
		a_3	STO	3			
3	Store B	b_1	STO	4			
		b_2	STO	5			
		b_3	STO	6			
4	Compute cross-product		f	PRGM	R/S		c_1
			R/S				c_2
			R/S				c_3
5	For new case, go to step 2.						

Example:

Let $A = (2, 5, 2)$

$B = (3, 3, -4).$

Solution:

$$A \times B = (-26, 14, -9)$$

ANGLE BETWEEN, NORM, AND DOT PRODUCT OF VECTORS

Let $\vec{a} = (a_1, a_2, \dots, a_n)$ and $\vec{b} = (b_1, b_2, \dots, b_n)$ be two vectors.

The norm of \vec{a} is denoted by $|\vec{a}|$ and is calculated by the following formula:

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + \dots + a_n^2}$$

similarly,

$$|\vec{b}| = \sqrt{b_1^2 + b_2^2 + \dots + b_n^2}$$

The dot product of \vec{a} and \vec{b} is denoted by $\vec{a} \cdot \vec{b}$ and is calculated by the following formula:

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \dots + a_n b_n$$

The angle between a and b is denoted by θ and is calculated by the following formula:

$$\theta = \cos^{-1} \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|} \right)$$

The angle is calculated in any angular mode. When calculated in degrees, decimal degrees are assumed.

DISPLAY		KEY ENTRY	DISPLAY		KEY ENTRY	REGISTERS	
LINE	CODE		LINE	CODE			
00			25			R ₀	$\sum a_i^2$
01	31	↑	26			R ₁	$\sum b_i^2$
02	14 02	$g x^2$	27			R ₂	$\sum a_i b_i$
03	23 51 01	STO + 1	28			R ₃	
04	22	R↓	29			R ₄	
05	21	$x \div y$	30			R ₅	
06	31	↑	31			R ₆	
07	14 02	$g x^2$	32			R ₇	
08	23 51 00	STO + 0	33				
09	22	R↓	34				
10	61	x	35				
11	23 51 02	STO + 2	36				
12	13 00	GTO 00	37				
13	24 02	RCL 2	38				
14	24 00	RCL 0	39				
15	24 01	RCL 1	40				
16	61	x	41				
17	14 02	$f\sqrt{x}$	42				
18	71	÷	43				
19	15 05	$g \cos^{-1}$	44				
20	13 00	GTO 00	45				
21			46				
22			47				
23			48				
24			49				

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Initialize		f	REG	f	PRGM	
3	Perform for $i = 1, \dots, n$:						
	Key in a_i and b_i	a_i	\uparrow				
		b_i	R/S				
4	Find norm of \vec{a}		RCL	0	f	\sqrt{x}	$ \vec{a} $
5	Find norm of \vec{b}		RCL	1	f	\sqrt{x}	$ \vec{b} $
6	Find $ \vec{a} \cdot \vec{b} $		RCL	2			$ \vec{a} \cdot \vec{b} $
7	Compute angle between \vec{a} and \vec{b}		GTO	13	R/S		θ

Example:

Let $A = (2, 5, 2)$
 $B = (3, 3, -4)$

Solution:

$$|\vec{a}| = 5.74$$

$$|\vec{b}| = 5.83$$

$$\vec{a} \cdot \vec{b} = 13.00$$

$$\theta = 67.16^\circ$$

SIMULTANEOUS EQUATIONS IN TWO UNKNOWNNS

Let $ax + by = e$

and $cx + dy = f$

be a system of two equations in two unknowns. Cramer's Rule is used to find the solution.

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{ed - bf}{ad - bc}$$

$$y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{af - ec}{ad - bc}$$

If $ad - bc = 0$ the calculator displays *Error*. In this case no solution or no unique solution exists.

DISPLAY		KEY ENTRY
LINE	CODE	
00		
01	24 03	RCL 3
02	24 05	RCL 5
03	61	x
04	24 02	RCL 2
05	24 06	RCL 6
06	61	x
07	41	-
08	24 01	RCL 1
09	24 05	RCL 5
10	61	x
11	24 02	RCL 2
12	24 04	RCL 4
13	61	x
14	41	-
15	23 00	STO 0
16	71	÷
17	74	R/S
18	24 01	RCL 1
19	24 06	RCL 6
20	61	x
21	24 03	RCL 3
22	24 04	RCL 4
23	61	x
24	41	-

DISPLAY		KEY ENTRY
LINE	CODE	
25	24 00	RCL 0
26	71	÷
27	13 00	GTO 00
28		
29		
30		
31		
32		
33		
34		
35		
36		
37		
38		
39		
40		
41		
42		
43		
44		
45		
46		
47		
48		
49		

REGISTERS
R ₀ ad - bc
R ₁ a
R ₂ b
R ₃ e
R ₄ c
R ₅ d
R ₆ f
R ₇

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Store constants	a	STO	1			
		b	STO	2			
		e	STO	3			
		c	STO	4			
		d	STO	5			
		f	STO	6			
3	Find x and y		f	PRGM	R/S		x
			R/S				y
4	For new case, go to step 2.						

Example:

$$5x - 3y = 12$$

$$2x + y = 9$$

Solution:

$$x = 3.55$$

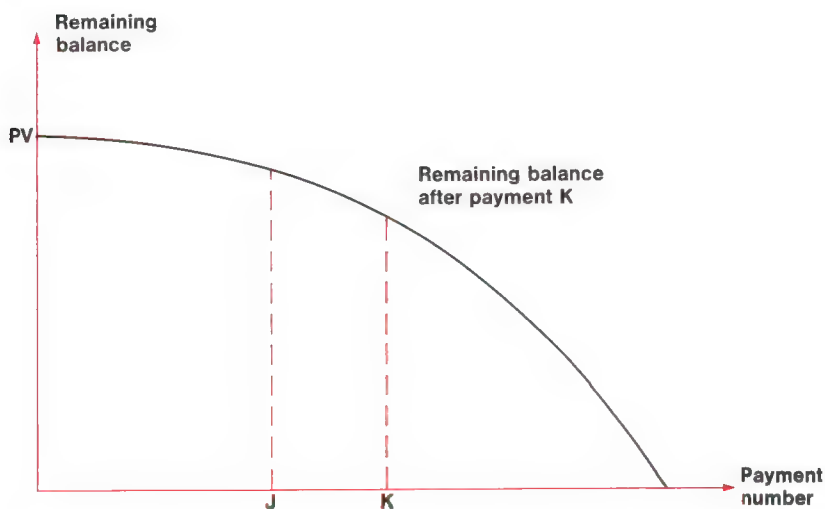
$$y = 1.91$$

CHAPTER 2 FINANCE

Because many of the finance programs have certain quantities in common, a word about these variables and the names used to refer to them may be helpful.

Five main variables recur in finance problems: n , i , PMT , PV , and FV . The first of these, n , denotes the total number of periods. The periodic interest rate i must be expressed in these programs as a decimal. Thus an annual interest rate of 6% is expressed as 0.06, which as a monthly rate would be $0.06/12 = 0.005$. PMT refers to the amount of the periodic payment. The present value, PV , is the value occurring at the beginning of the first period, while the future value, FV , is the value at the end of the last period.

MORTGAGE LOAN ACCUMULATED INTEREST/REMAINING BALANCE



As one enters into the realm of financial calculations, one of the most striking revelations is how much of the repayment of a loan goes to interest. A new homeowner, for example, sends off his first monthly installment of \$220.13 toward repayment of a 30-year, \$30,000 mortgage assumed at 8% annual interest. With a proud sigh and a swelling chest, the homeowner mentally checks \$220 off the \$30,000 and figures he's well on his way. Right? Well, not quite. In fact, \$200 of that payment will go to interest, and only \$20.13 to reducing the principal of the loan.

This program will allow the user to calculate the amount paid to interest, for one payment or over a number of payments, as well as the amount of principal still unpaid, i.e., the remaining balance. The user must input the following values: the initial amount of the loan, the periodic interest rate, and the periodic payment amount. He must then key in a beginning payment number, J , and an ending payment number, K . The program will compute the accumulated interest charge from payment J through payment K , inclusive, and the balance remaining after payment K . If one wishes to find the amount of interest paid in a single payment, he can simply set $K = J$.

The program can also be used to generate a limited amortization schedule showing the balance remaining after successive payments. This can be done by leaving $J = 1$ and increasing K by 1 at each iteration. Outputs will be the total amount paid to interest over the first K payments, and the balance remaining after payment K .

Equations:

$$BAL_K = \frac{1}{(1+i)^{-K}} \left[PMT \frac{(1+i)^{-K} - 1}{i} + PV \right]$$

$$Int_{J-K} = BAL_K - BAL_{J-1} + (K - J + 1) PMT$$

where BAL_n = remaining balance after payment n

Int_{J-K} = accumulated interest, payments J through K

PV = initial loan amount

PMT = periodic payment amount

i = periodic interest rate

Notes:

1. The periodic interest rate i must be entered as a decimal. For example, for monthly payments with an annual interest rate of 9%, the periodic interest rate should be input as $i = \frac{.09}{12} = 0.0075$.
2. The use of this program is not restricted to mortgage loans, but applies equally well to any loan which is being repaid with equal periodic payments.

Programming Remarks:

In many finance programs, the expressions $(1+i)$ and $(1+i)^n$ are used several times per program. It is often simpler to calculate the quantity once and then store it for later use, rather than calculate it anew each time. In this program, the values of $(1+i)^{-K}$ and $(1+i)^{-J}$ are calculated once and then stored in R_7 , thus saving both program steps and execution time. The same principle, of course, applies to other expressions in other problems.

DISPLAY		KEY ENTRY	X	Y	Z	T	COMMENTS	REGISTERS
LINE	CODE							
00								
01	24 01	RCL 1	i				Calculate BAL_K	R_0
02	01	1	1	i				
03	51	+	$1+i$					R_1
04	24 05	RCL 5	K	$1+i$				
05	32	CHS	-K	$1+i$				
06	14 03	$f y^x$	$(1+i)^{-K}$					R_2 PMT
07	23 07	STO 7	$(1+i)^{-K}$					
08	01	1	1	$(1+i)^{-K}$				
09	41	-	$(1+i)^{-K-1}$					R_3 PV
10	24 01	RCL 1	i	$(1+i)^{-K-1}$				
11	71	\div	s				Let $s = [(1+i)^{-K-1}] \div i$	
12	24 02	RCL 2	PMT	s				R_4 J
13	61	x	PMT s					
14	24 03	RCL 3	PV	PMT s				R_5 K
15	51	+	PMT s + PV					
16	24 07	RCL 7	$(1+i)^{-K}$	PMT s + PV				
17	71	\div	BAL_K					
18	23 06	STO 6	BAL_K					R_6 BAL_K
19	24 01	RCL 1	i	BAL_K			Calculate BAL_{J-1}	
20	01	1	1	i	BAL_K			
21	51	+	$(1+i)$	BAL_K				
22	24 04	RCL 4	J	$(1+i)$	BAL_K			R_7 $(1+i)^{-n}$
23	01	1	1	J	$(1+i)$	BAL_K		
24	41	-	J - 1	$(1+i)$	BAL_K	BAL_K		
25	32	CHS	-(J - 1)	$(1+i)$	BAL_K	BAL_K		
26	14 03	$f y^x$	$(1+i)^{-(J-1)}$	BAL_K	BAL_K	BAL_K		
27	23 07	STO 7	$(1+i)^{1-J}$	BAL_K	BAL_K	BAL_K		
28	01	1	1	$(1+i)^{1-J}$	BAL_K	BAL_K		
29	41	-	$(1+i)^{1-J-1}$	BAL_K	BAL_K	BAL_K		
30	24 01	RCL 1	i	$(1+i)^{1-J-1}$	BAL_K	BAL_K		
31	71	\div	s	BAL_K	BAL_K	BAL_K	Let $s = [(1+i)^{1-J-1}] \div i$	
32	24 02	RCL 2	PMT	s	BAL_K	BAL_K		
33	61	x	PMT s	BAL_K	BAL_K	BAL_K		
34	24 03	RCL 3	PV	PMT s	BAL_K	BAL_K		
35	51	+	PMT s + PV	BAL_K	BAL_K	BAL_K		
36	24 07	RCL 7	$(1+i)^{1-J}$	PMT s + PV	BAL_K	BAL_K		
37	71	\div	BAL_{J-1}	BAL_K	BAL_K	BAL_K		
38	41	-	Diff	BAL_K	BAL_K	BAL_K	Diff = $BAL_K - BAL_{J-1}$	
39	24 05	RCL 5	K	Diff	BAL_K	BAL_K	K - J + 1 gives no. PMT's	
40	24 04	RCL 4	J	K	Diff	BAL_K	from J through K	
41	41	-	K - J	Diff	BAL_K	BAL_K		
42	01	1	1	K - J	Diff	BAL_K		
43	51	+	K - J + 1	Diff	BAL_K	BAL_K		
44	24 02	RCL 2	PMT	m	Diff	BAL_K	m = K - J + 1	
45	61	x	m PMT	Diff	BAL_K	BAL_K	m PMT is \$ paid, J-K	
46	51	+	Int $_{J-K}$	BAL_K	BAL_K	BAL_K	Display Int $_{J-K}$	
47	74	R/S	Int $_{J-K}$	BAL_K	BAL_K	BAL_K		
48	21	$x \div y$	BAL_K	Int $_{J-K}$	BAL_K	BAL_K	Display BAL_K	
49	13 00	GTO 00	BAL_K	Int $_{J-K}$	BAL_K	BAL_K		

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Store the following variables:						
	Periodic interest (decimal)	i	STO	1			
	Periodic payment	PMT	STO	2			
	Initial loan amount	PV	STO	3			
	Starting payment number	J	STO	4			
	Ending payment number	K	STO	5	f	PRGM	
3	Compute accumulated interest from payments J through K.						Int _{J,K}
4	Display remaining balance after payment K						BAL _K
5	To change any variable, store the new value in the appropriate register and go to step 3.						

Example:

A mortgage is arranged so that the first payment is made at the end of October, 1974 (i.e., October is payment period 1). It is a \$25,000 loan at 8% with monthly payments of \$200. What is the accumulated interest for 1974 (periods 1–3) and for 1975 (periods 4–15) and what balance remains at the end of each year? Also, generate a schedule of interest paid and remaining balance for the first 5 years of the mortgage (periods 12, 24, 36, 48, 60).

Solution:

(Notice that i must be entered as a decimal, monthly rate.)

.08 \uparrow 12 \div STO 1 200 STO 2 25000 STO 3 1

STO 4 3 STO 5 f PRGM R/S \longrightarrow 499.33
(interest paid in 1974)

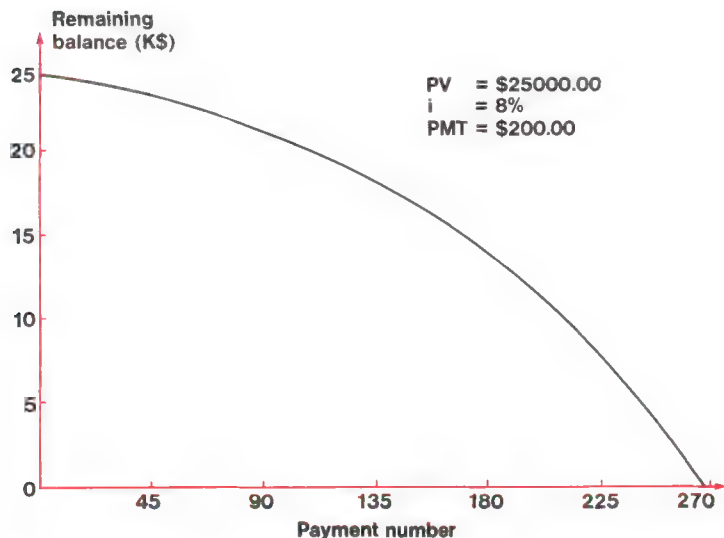
R/S \longrightarrow 24899.33
(remaining balance at end of 1974)

4 STO 4 15 STO 5 R/S \longrightarrow 1976.65
(interest paid in 1975)

R/S \longrightarrow 24475.98
(remaining balance at end of 1975)

Now, generate the amortization schedule:

1	STO	4	12	STO	5	R/S	→ 1985.00	(interest thru 1 st year)
	R/S						→ 24585.00	(remaining balance after 1 st year)
24	STO	5		R/S			→ 3935.56	(interest thru 2 nd year)
	R/S						→ 24135.56	(remaining balance after 2 nd year)
36	STO	5		R/S			→ 5848.81	(interest thru 3 rd year)
	R/S						→ 23648.81	(remaining balance after 3 rd year)
48	STO	5		R/S			→ 7721.67	(interest thru 4 th year)
	R/S						→ 23121.67	(remaining balance after 4 th year)
60	STO	5		R/S			→ 9550.77	(interest thru 5 th year)
	R/S						→ 22550.77	(remaining balance after 5 th year)



MORTGAGE LOAN PAYMENT, PRESENT VALUE, NUMBER OF PERIODS



For a loan which is being repaid with equal periodic payments, this program will calculate the payment amount, the present value, or the number of periods of the loan, given the periodic interest rate and the two other variables.

Remember that the periodic interest rate i must be expressed as a decimal, e.g., 6% is represented as 0.06.

The equations used are as follows:

$$PMT = PV \left[\frac{i}{1 - (1 + i)^{-n}} \right] \quad PV = PMT \left[\frac{1 - (1 + i)^{-n}}{i} \right]$$

$$n = - \frac{\ln(1 - i PV / PMT)}{\ln(1 + i)}$$

DISPLAY		KEY ENTRY
LINE	CODE	
00		
01	01	1
02	24 02	RCL 2
03	01	1
04	51	+
05	24 01	RCL 1
06	32	CHS
07	14 03	f y ^x
08	41	-
09	24 02	RCL 2
10	21	x<=y
11	71	÷
12	24 04	RCL 4
13	61	x
14	13 00	GTO 00
15	01	1
16	24 02	RCL 2
17	01	1
18	51	+
19	24 01	RCL 1
20	32	CHS
21	14 03	f y ^x
22	41	-
23	24 02	RCL 2
24	71	÷

DISPLAY		KEY ENTRY
LINE	CODE	
25	24 03	RCL 3
26	61	x
27	13 00	GTO 00
28	01	1
29	24 04	RCL 4
30	24 03	RCL 3
31	71	÷
32	24 02	RCL 2
33	61	x
34	41	-
35	14 07	f LN
36	24 02	RCL 2
37	01	1
38	51	+
39	14 07	f LN
40	71	÷
41	32	CHS
42	13 00	GTO 00
43		
44		
45		
46		
47		
48		
49		

REGISTERS	
R ₀	
R ₁ n	
R ₂ i	
R ₃ PMT	
R ₄ PV	
R ₅	
R ₆	
R ₇	

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	For payment	n	STO	1			
		i	STO	2			
		PV	STO	4			
			f	PRGM	R/S		PMT
3	For present value	n	STO	1			
		i	STO	2			
		PMT	STO	3			
			GTO	15	R/S		PV
4	For number of payments	i	STO	2			
		PMT	STO	3			
		PV	STO	4			
			GTO	28	R/S		n
5	For new case, go to step 2, 3, or 4.						

Examples:

- What monthly payment is required to amortize a \$3000 loan at 9.5% (.095) in 36 months?
- You are willing to pay \$175 per month for 24 months on a 9.5% loan. How much can you borrow?
- How many months will it take to pay off a \$4000 loan if your monthly payment is \$200 and the annual interest rate is 9.5%?

Solutions:

Divide 0.095 by 12 to find the monthly interest rate expressed as a decimal.

- \$96.10
- \$3811.43
- 21.86 months

MORTGAGE LOAN INTEREST RATE



This program will calculate the interest rate on a loan with equal periodic payments. The user must specify the number of periods, the present value or initial loan amount, and the payment amount.

The program performs an iterative solution for i using Newton's method:

$$i_{k+1} = i_k - \frac{f(i_k)}{f'(i_k)}$$

where

$$f(i) = \frac{1 - (1 + i)^{-n}}{i} - \frac{PV}{PMT}$$

The initial guess for i is given by

$$i_0 = \frac{PMT}{PV} - \frac{PV}{n^2 PMT}$$

DISPLAY		KEY ENTRY
LINE	CODE	
00		
01	24 03	RCL 3
02	31	↑
03	15 22	g 1/x
04	21	x↔y
05	24 01	RCL 1
06	15 02	g x ²
07	71	÷
08	41	-
09	23 02	STO 2
10	24 03	RCL 3
11	24 02	RCL 2
12	61	x
13	01	1
14	24 02	RCL 2
15	01	1
16	51	+
17	24 01	RCL 1
18	32	CHS
19	14 03	f y ^x
20	23 05	STO 5
21	41	-
22	41	-
23	24 01	RCL 1
24	24 02	RCL 2

DISPLAY		KEY ENTRY
LINE	CODE	
25	15 22	g 1/x
26	01	1
27	51	+
28	71	÷
29	01	1
30	51	+
31	24 05	RCL 5
32	61	x
33	01	1
34	41	-
35	24 02	RCL 2
36	71	÷
37	71	÷
38	23 51 02	STO + 2
39	15 03	g ABS
40	33	EEX
41	06	6
42	32	CHS
43	14 41	f x<y
44	13 10	GTO 10
45	24 02	RCL 2
46	13 00	GTO 00
47		
48		
49		

REGISTERS	
R ₀	
R ₁ n	
R ₂ i	
R ₃ PV/PMT	
R ₄ (1 + i) ⁻ⁿ	
R ₅	
R ₆	
R ₇	

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Store number of payments	n	STO	1			
3	Key in present value and pay- ment amount	PV	↑				
		PMT	÷	STO	3		PV/PMT
4	Compute interest		f	PRGM	R/S		i (decimal)
			EEX	2	x		i (%)
5	For new case, go to step 2.						

Example:

You recently obtained a \$2500 car loan for 36 months. If your monthly payment is \$86.67, what is the annual percentage rate?

Solution:

15.01%

COMPOUND AMOUNT



This program applies to an amount of principal that has been placed into an account and compounded periodically, with no further deposits. The important variables in this case are the number of compounding periods n , the periodic interest rate i , the principal or present value PV , the future value of the account FV , and the amount of interest accrued I . Any of these may be calculated from the others by these formulas:

$$n = \frac{\ln(FV/PV)}{\ln(1+i)} \quad i = \left(\frac{FV}{PV} \right)^{1/n} - 1 \quad PV = FV(1+i)^{-n}$$

$$FV = PV(1+i)^n \quad I = PV[(1+i)^n - 1]$$

DISPLAY		KEY ENTRY
LINE	CODE	
00		
01	24 05	RCL 5
02	24 04	RCL 4
03	71	÷
04	14 07	f LN
05	24 02	RCL 2
06	01	1
07	51	+
08	14 07	f LN
09	71	÷
10	13 00	GTO 00
11	24 05	RCL 5
12	24 04	RCL 4
13	71	÷
14	24 01	RCL 1
15	15 22	g 1/x
16	14 03	f y ^x
17	01	1
18	41	-
19	13 00	GTO 00
20	24 02	RCL 2
21	01	1
22	51	+
23	24 01	RCL 1
24	32	CHS

DISPLAY		KEY ENTRY
LINE	CODE	
25	14 03	f y ^x
26	24 05	RCL 5
27	61	x
28	13 00	GTO 00
29	24 02	RCL 2
30	01	1
31	51	+
32	24 01	RCL 1
33	14 03	f y ^x
34	24 04	RCL 4
35	61	x
36	13 00	GTO 00
37	24 02	RCL 2
38	01	1
39	51	+
40	24 01	RCL 1
41	14 03	f y ^x
42	01	1
43	41	-
44	24 04	RCL 4
45	61	x
46	13 00	GTO 00
47		
48		
49		

REGISTERS	
R ₀	
R ₁ n	
R ₂ i	
R ₃	
R ₄ PV	
R ₅ FV	
R ₆	
R ₇	

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	To compute number of periods	i (decimal)	STO	2			
		PV	STO	4			
		FV	STO	5			
			f	PRGM	R/S		n
3	To compute periodic interest rate	n	STO	1			
		PV	STO	4			
		FV	STO	5			
			GTO	11	R/S		i (decimal)
4	To compute principal	n	STO	1			
		i (decimal)	STO	2			
		FV	STO	5			
			GTO	20	R/S		PV
5	To compute future value	n	STO	1			
		i (decimal)	STO	2			
		PV	STO	4			
			GTO	29	R/S		FV
6	To compute accrued interest	n	STO	1			
		i (decimal)	STO	2			
		PV	STO	4			
			GTO	37	R/S		I
7	For new case, go to step 2, 3, 4, 5, or 6.						

Examples:

- Assuming an annual inflation rate of 10%, how long will it take prices to double? (Suggestion: let $PV = 1$, $FV = 2$)
- Find the rate of return on \$1000 compounded quarterly if it amounts to \$1500 in 5 years.
- How much will you need to invest today at $5\frac{3}{4}\%$ compounded quarterly to have \$3000 in 5 years?
- What is the future value of \$2000 invested at $5\frac{3}{4}\%$ compounded quarterly for 4 years (16 quarters)?
- How much interest do you receive on \$1500 deposited for 10 years if interest at $5\frac{1}{2}\%$ is compounded annually?

Solutions:

1. 7.27 years
2. .0205 quarterly = 8.19% annually
3. \$2255.02 ($i = 0.0575/4$)
4. \$2513.08 ($i = 0.0575/4$)
5. \$1062.22 ($i = 0.055$)

PERIODIC SAVINGS **PAYMENT, FUTURE VALUE, NUMBER OF PERIODS**



This program calculates payment, future value, or number of time periods for a schedule of periodic payments into a savings account, given the interest rate and two of the three other variables. Remember that i must be input as a decimal, e.g., 6% is expressed as 0.06.

Then n , PMT , or FV may be calculated from the following formulas:

$$n = \frac{\ln \left[\frac{FV i}{PMT} + (1 + i) \right]}{\ln (1 + i)} - 1 \qquad PMT = \frac{FV i}{(1 + i)^{n+1} - (1 + i)}$$

$$FV = \frac{PMT}{i} \left[(1 + i)^{n+1} - (1 + i) \right]$$

DISPLAY		KEY ENTRY
LINE	CODE	
00		
01	24 02	RCL 2
02	24 05	RCL 5
03	61	x
04	24 03	RCL 3
05	71	÷
06	24 02	RCL 2
07	01	1
08	51	+
09	23 00	STO 0
10	51	+
11	14 07	f LN
12	24 00	RCL 0
13	14 07	f LN
14	71	÷
15	01	1
16	41	-
17	13 00	GTO 00
18	24 05	RCL 5
19	24 00	RCL 2
20	61	x
21	24 02	RCL 2
22	01	1
23	51	+
24	71	÷

DISPLAY		KEY ENTRY
LINE	CODE	
25	14 73	f LASTx
26	24 01	RCL 1
27	14 03	f y ^x
28	01	1
29	41	-
30	71	÷
31	13 00	GTO 00
32	24 03	RCL 3
33	24 02	RCL 2
34	01	1
35	51	+
36	61	x
37	14 73	f LASTx
38	24 01	RCL 1
39	14 03	f y ^x
40	01	1
41	41	-
42	61	x
43	24 02	RCL 2
44	71	÷
45	13 00	GTO 00
46		
47		
48		
49		

REGISTERS	
R ₀	(1 + i)
R ₁	n
R ₂	i
R ₃	PMT
R ₄	
R ₅	FV
R ₆	
R ₇	

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	To compute number of payments	i (decimal)	STO	2			
		PMT	STO	3			
		FV	STO	5			
			f	PRGM	R/S		n
3	To compute periodic payment amount	n	STO	1			
		i (decimal)	STO	2			
		FV	STO	5			
			GTO	18	R/S		PMT
4	To compute future value	n	STO	1			
		i (decimal)	STO	2			
		PMT	STO	3			
			GTO	32	R/S		FV
5	For new case, go to step 2, 3, or 4.						

Examples:

- How long will it take to save \$15,000 if you are making quarterly deposits of \$400 at 6% annual interest?
- You will need \$10,000 in 7 years. How large a monthly payment do you need to make if the annual interest rate is 6 1/2%?
- How much money will a person have if he deposits \$150 at the end of each month for a period of 3 years? He receives 6% annual interest.

Solutions:

- 29.62 quarters or 7.40 years ($i = .06/4$)
- \$93.82 ($n = 84, i = .065/12$)
- \$5929.92 ($n = 36, i = .06/12$)

DISCOUNTED CASH FLOW NET PRESENT VALUE, INTERNAL RATE OF RETURN

The primary purpose of this program is to compute the net present value of a series of cash flows. In general, an initial investment V_0 is made in some enterprise which is expected to bring in periodic cash flows C_1, C_2, \dots, C_n . Given a discount rate i , which must be entered as a decimal, then for each cash flow C_k , the program will compute the net present value at period k , NPV_k . A negative value for NPV_k indicates that the enterprise has not yet been profitable. A positive NPV_k means that the enterprise has been profitable, to the extent that a rate of return i on the original investment has been exceeded.

The program may also be used iteratively to calculate an internal rate of return. The objective here is to find the discount rate i which will make the final net present value, NPV_n , equal to zero. The procedure, then, is to store V_0 and a first guess at the rate of return i , input the cash flows C_1 through C_n ; and thus find NPV_n . If NPV_n is negative, the estimated rate of return was too high; if NPV_n is positive, the estimate for i was too low. Adjust the estimate for i accordingly, store the new i , and input the cash flows again. Inspect the new value of NPV_n to obtain a new estimate for i and repeat the process. The entire procedure is repeated until NPV_n is zero, or very close to it. The last value of i input is then regarded as the internal rate of return.

Each figure for net present value is found by

$$NPV_k = -V_0 + \sum_{j=1}^k \frac{C_j}{(1+i)^j}$$

DISPLAY		KEY ENTRY
LINE	CODE	
00		
01	24 01	RCL 1
02	01	1
03	23 04	STO 4
04	51	+
05	23 02	STO 2
06	71	÷
07	24 00	RCL 0
08	41	-
09	24 04	RCL 4
10	14 74	f PAUSE
11	21	$x \div y$
12	23 03	STO 3
13	74	R/S
14	24 02	RCL 2
15	24 04	RCL 4
16	01	1
17	51	+
18	23 04	STO 4
19	14 03	f y ^x
20	71	÷
21	24 03	RCL 3
22	51	+
23	13 09	GTO 09
24		

DISPLAY		KEY ENTRY
LINE	CODE	
25		
26		
27		
28		
29		
30		
31		
32		
33		
34		
35		
36		
37		
38		
39		
40		
41		
42		
43		
44		
45		
46		
47		
48		
49		

REGISTERS
R ₀ V ₀
R ₁ i
R ₂ (1 + i)
R ₃ NPV _k
R ₄ k
R ₅
R ₆
R ₇

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Store initial investment and discount rate	V ₀	STO	0			
		i (decimal)	STO	1	f	PRGM	
3	Perform for k = 1, ..., n:						
	Input C _k and compute NPV _k	C _k	R/S				(k)
							NPV _k
4	For new case, go to step 2.						

Example:

You have been offered an investment opportunity for \$150,000 at a capital cost of 10% after taxes. Based on the following cash flows, will this investment be profitable?

Year	Cash Flow
1	\$30,000
2	26,300
3	50,000
4	55,600
5	45,200

Solutions:

Remember to enter i as 0.10.

$$NPV_1 = -\$122,727.27$$

$$NPV_2 = -\$100,991.74$$

$$NPV_3 = -\$63,426.00$$

$$NPV_4 = -\$25,450.45$$

$$NPV_5 = \$2,615.20$$

Since C_5 is positive the cash flow is profitable to the extent that the cost of capital is 10%.

CALENDAR DAY OF THE WEEK DAYS BETWEEN TWO DATES

This program will compute the day of the week for a given date, or the number of days between two dates, for any dates from March 1, 1700, to February 28, 2100. The program works by assigning the number 1 to March 1, 1700, and a corresponding number to each succeeding day. When computing day of the week, a 0 represents Sunday, 1 Monday, 2 Tuesday, etc.

Thus for month m , day d , year y , the number N assigned to that date is

$$N(m, d, y) = [365.25 g(y, m)] + [30.6 f(m)] + D - 621049$$

where

$$g(y, m) = \begin{cases} y - 1 & \text{if } m = 1 \text{ or } 2 \\ y & \text{if } m > 2 \end{cases} \quad \text{and } f(m) = \begin{cases} m + 13 & \text{if } m = 1 \text{ or } 2 \\ m + 1 & \text{if } m > 2 \end{cases}$$

$[m]$ represents the integer function, $\boxed{f} \boxed{\text{INT}}$. E.g., $[6.34] = 6$.

Note:

For days from March 1, 1700, to February 28, 1800, 2 days must be added to the value for N calculated by the program. For days from March 1, 1800, to February 28, 1900, 1 day must be added.

DISPLAY		KEY ENTRY
LINE	CODE	
00		
01	03	3
02	24 01	RCL 1
03	14 41	f x<y
04	13 09	GTO 09
05	01	1
06	51	+
07	24 03	RCL 3
08	13 15	GTO 15
09	01	1
10	03	3
11	51	+
12	24 03	RCL 3
13	01	1
14	41	-
15	03	3
16	06	6
17	05	5
18	73	.
19	02	2
20	05	5
21	61	x
22	14 01	f INT
23	21	x<2y
24	03	3

DISPLAY		KEY ENTRY
LINE	CODE	
25	00	0
26	73	.
27	06	6
28	61	x
29	14 01	f INT
30	51	+
31	24 02	RCL 2
32	51	+
33	06	6
34	02	2
35	01	1
36	00	0
37	04	4
38	09	9
39	41	-
40	74	R/S
41	07	7
42	71	÷
43	15 01	g FRAC
44	07	7
45	61	x
46	13 00	GTO 00
47		
48		
49		

REGISTERS	
R ₀	
R ₁ Month	
R ₂ Day	
R ₃ Year	
R ₄	
R ₅	
R ₆	
R ₇ Temporary	

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Store month	m	STO	1			
	day	d	STO	2			
	year	y	STO	3			
3	Compute $N(m, d, y)$		f	PRGM	R/S		$N(m, d, y)$
4	For day of week, go to step 8						
5	For days between dates, store						
	first N		STO	7			
6	Repeat steps 2 and 3 for second						
	date, then		RCL	7	-		# Days
7	For new case, go to step 2.						
8	For day of week (0 = Sunday)		R/S				Day (0,..., 6)
9	For new case, go to step 2.						

Examples:

- What day of the week was July 4, 1776?
- Find the number of days between March 27, 1948, and April 7, 1975.

Solutions:

- Thursday (4). (Remember to add 2 days.)
- 9872.

Notes

[illegible]

CHAPTER 3 GAMES

MOON LANDING SIMULATOR

Imagine for a moment the difficulties involved in landing a rocket on the moon with a strictly limited fuel supply. You're coming down tail-first, free-falling toward a hard rock surface. You'll have to ignite your rockets to slow your descent; but if you burn too much too soon, you'll run out of fuel 100 feet up, and then you'll have nothing to look forward to but cold eternal moon dust coming faster every second. The object, clearly, is to space your burns just right so that you will alight on the moon's surface with no downward velocity.

The game starts off with the rocket descending at a velocity of 50 feet/sec from a height of 500 feet. The velocity and height are shown in a combined display as -50.0500, the height appearing to the right of the decimal point and the velocity to the left, with a negative sign on the velocity to indicate downward motion. If a velocity is ever displayed with no fractional part, for example, -15., it means that you have crashed at a speed of 15 feet/sec. In game terms, this means that you have lost; in real-life, it signifies an even less favorable outcome.

You will start the game with 120 units of fuel. You may burn as much or as little of your available fuel as you wish at each step of your descent; burns of zero are quite common. A burn of 5 units will just cancel gravity and hold your speed constant. Any burn over 5 will act to change your speed in an upward direction. You must take care, however, not to burn more fuel than you have; for if you do, no burn at all will take place, and you will free-fall to your doom! The final velocity shown will be your impact velocity (generally rather high). You may display your remaining fuel at any time by recalling R_2 .

Equations:

We don't want to get too specific, because that would spoil the fun of the game; but rest assured that the program is solidly based on some old friends from Newtonian physics:

$$x = x_0 + v_0 t + \frac{1}{2} a t^2 \qquad v = v_0 + a t \qquad v^2 = v_0^2 + 2 a x$$

where x , v , a , and t are distance, velocity, acceleration, and time.

Notes:

1. If you crash before running out of fuel, the crash velocity shown will be the velocity before the burn, rather than the impact velocity.
2. Use only integer values for burns. Any decimal entry will cause an error in the display for V.X.

Programming Remarks:

An interesting feature of this program is the simultaneous display of both speed and altitude (V,X), as for example, -50.0500. This is accomplished by storing the speed V and the altitude X in their normal form (-50.00, 500.00), then dividing X by 10,000 (10^4) before combining them. An additional subtlety involves the question of the sign of V, and whether ($X/10^4$) is to be added to or subtracted from V. For example, if $V = -50$ and $X = 500$, we should subtract: $V - (X/10^4)$, in order to generate a display of -50.0500. But if $V = 10$ and $X = 50$, we should add: $V + (X/10^4)$ in order to display 10.0050. Inspection of the program listing, lines 2 through 12, will reveal how a conditional branch was used to resolve the dilemma.

DISPLAY		KEY ENTRY	X	Y	Z	T	COMMENTS	REGISTERS
LINE	CODE							
00								R 0 X
01	14 11 04	f FIX 4					Four-place display	
02	24 00	RCL 0	X				Form display V X	
03	33	EEX	1. 00	X				R 1 V
04	04	4	1. 04	X				
05	71	÷	$X/10^4$				Divide X by 10,000	
06	24 01	RCL 1	V	$X/10^4$				R 2 Fuel
07	15 41	g x<0	V	$X/10^4$			Is V negative?	
08	13 11	GTO 11	V	$X/10^4$			Yes, branch	
09	51	+	$V + X/10^4$				No, add V and X	R 3 Acceleration
10	13 13	GTO 13	$V + X/10^4$					
11	21	$x \geq y$	$X/10^4$	V			$V < 0$, add V and -X	R 4
12	41	-	$V - X/10^4$					
13	74	R/S	V X				V, X is $V \pm (X/10^4)$	R 5
14	24 02	RCL 2	F	B			Burn B has been input	
15	14 41	f x<y	F	B			Burn > Fuel?	
16	13 34	GTO 34	F	B			Yes, prepare to crash	
17	22	R↓	B			F	No, update A, X, V	
18	23 41 02	STO - 2	B			F	Subtract burn from fuel	R 6
19	05	5	5	B			5 units cancels gravity	
20	41	-	$B - 5$				Acceleration = $B - 5$	R 7
21	23 03	STO 3	A					
22	02	2	2	A				
23	71	÷	$A/2$					
24	24 00	RCL 0	X	$A/2$				
25	51	+	$X + A/2$					
26	24 01	RCL 1	V	$X + A/2$				
27	51	+	$X + V + A/2$				New altitude $X = X + V + A/2$	
28	23 00	STO 0	X					
29	15 41	g x<0	X				Is X below ground?	
30	13 44	GTO 44	X				Yes, you've crashed	
31	24 03	RCL 3	A	X			No, update V	
32	23 51 01	STO + 1	A	X			New velocity: $V \leftarrow V + A$	
33	13 02	GTO 02	A	X			Display V,X	
34	24 01	RCL 1	V				All fuel gone, show	
35	15 02	g x ²	V^2				crash velocity as	
36	24 00	RCL 0	X	V^2			$V = (V^2 + 2gX)^{1/2}$	
37	01	1	1	X	V^2		where g = gravity = 5	
38	00	0	10	X	V^2			
39	61	x	10 X	V^2				
40	51	+	$V^2 + 10 X$					
41	14 02	f √x	V					
42	32	CHS	V				Show crash V down	
43	23 01	STO 1	V					
44	24 01	RCL 1	V				Come here from line 30	
45	14 11 00	f FIX 0	V				Display integer V to	
46	13 00	GTO 00	V				show crash	
47								
48								
49								

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Initialize	X	500	STO	0		500.00
		V	50	CHS	STO	1	-50.00
		Fuel	120	STO	2		120.00
3	Display initial V.X		f	PRGM	R/S		-50.0500
4	Key in burn, compute new speed and distance	Burn	R/S				V.X
5	Perform step 4 till you land or crash						
6	To see remaining fuel at any time		RCL	2			Fuel
7	To display speed and distance at any time		f	PRGM	R/S		V.X
8	To start a new game, go to step 2.						

Example:

500 **STO** **0** 50 **CHS** **STO** **1** 120 **STO** **2**

f **PRGM** **R/S** → -50.0500

0 **R/S** → -55.0448

5 **R/S** → -55.0393

(note constant V when burn = 5)

30 **R/S** → -30.0350

0 **R/S** → -35.0318

0 **R/S** → -40.0280

0 **R/S** → -45.0238

0 **R/S** → -50.0190

RCL **2** → 85.0000
(remaining fuel)

f **PRGM** **R/S** → -50.0190
(display V.X again)

10 **R/S** → -45.0143

0 **R/S** → -50.0095

RCL **2** → 75.0000

10 **R/S** → -45.0048

25 **R/S** → -25.0013

20 **R/S** → -25.

Oops.

NIMB

The game of Nimb begins with a collection of N objects, or as the calculator plays it, with the positive number N . Each player alternately subtracts one, two, or three from the total until only one is left. The player forced to take the last one loses.

To begin the game, you must tell the machine how many objects to start with, i.e., the value of N . A reasonable number is 15. After each move the machine will display the remaining total. A negative sign indicates that it is the user's move next, while a positive display indicates that it is the HP-25's move.

As the challenger you are allowed to make the first move. It is possible to win but of course the HP-25 is a master player: it will not let you make an error and win. (Not, that is, unless you cheat and take a number other than 1, 2, or 3—a contingency so far beyond the realm of the HP-25's naive faith in human-kind that the unsuspecting calculator has no way of knowing if you do or don't.)

DISPLAY		KEY ENTRY
LINE	CODE	
00		
01	01	1
02	23 02	STO 2
03	22	R↓
04	23 41 00	STO - 0
05	24 00	RCL 0
06	15 71	g x=0
07	13 41	GTO 41
08	23 61 02	STO x 2
09	24 02	RCL 2
10	74	R/S
11	21	x ∇ z y
12	15 51	g x \geq 0
13	13 16	GTO 16
14	21	x ∇ z y
15	13 01	GTO 01
16	01	1
17	32	CHS
18	23 02	STO 2
19	00	0
20	23 01	STO 1
21	24 01	RCL 1
22	03	3
23	14 71	f x=y
24	13 39	GTO 39

DISPLAY		KEY ENTRY
LINE	CODE	
25	01	1
26	23 51 01	STO + 1
27	32	CHS
28	24 00	RCL 0
29	51	+
30	24 01	RCL 1
31	41	-
32	04	4
33	71	÷
34	15 01	g FRAC
35	15 61	g x \neq 0
36	13 21	GTO 21
37	24 01	RCL 1
38	13 04	GTO 04
39	01	1
40	13 04	GTO 04
41	24 02	RCL 2
42	15 41	g x<0
43	13 46	GTO 46
44	24 02	RCL 3
45	13 00	GTO 00
46	24 04	RCL 4
47	14 11 01	f FIX 1
48	13 00	GTO 00
49		

REGISTERS	
R ₀	Total
R ₁	Machine move
R ₂	± Total
R ₃	55178
R ₄	3507.1
R ₅	
R ₆	
R ₇	

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Initialize	55178	STO	3			
		3507.1	STO	4	f	PRGM	
3	Store total number of objects						
	(usually 15) and set display	N	STO	0	CHS	f	
			FIX	0			-N.
4	If number in display is negative,						
	key in your move	Your move	R/S				+ Total
5	If number in display is positive,						
	let HP-25 move		R/S				- Total
6	Perform steps 4 and 5 until game						
	is over						
7	At end of game, turn calculator						
	upside down to read message						
8	For another game, go to step 3.						

Example:

Perform the initialization with $N = 15$.

User takes 3.

3 R/S —————→ 12.

R/S —————→ -9.

HP-25 takes 3.

User takes 2.

2 R/S —————→ 7.

R/S —————→ -5.

HP-25 takes 2.

User takes 3.

3 R/S —————→ 2.

R/S —————→ -1.

HP-25 takes 1.

User takes last 1.

1 R/S —————→ 55178.

Turn calculator upside down for message (BLISS).

TEACH ARITHMETIC

We at Hewlett-Packard feel that the hand-held calculator, far from threatening the traditional tenets of a sound mathematics education, may be used creatively to reinforce learning in such areas as arithmetic, algebra, geometry, trigonometry, calculus, and numerical analysis. This program, which is designed to be used in teaching children the four operations of elementary arithmetic ($+$, $-$, \times , \div), demonstrates some of the (largely unexplored) potential of the HP-25 as an educational tool.

The basic flow of the program is to pose a problem in arithmetic, check the answer that the user keys in against the correct answer, and then do one of two things: if the user's answer was correct, the program will go on to pose a new problem; if the keyed-in answer was wrong, the program restates the original problem to give the learner a second chance.

To run the program, the user must store a value called Max in R_0 . This tells the program not to use any numbers as large as Max in its problems. If you specify a Max of 12, for example, then all the problems will deal with numbers between 0 and 11. The user must then store in R_1 a "seed" s , a number between 0 and 1, which will determine the sequence of problems that will appear. Different seeds generate different problems, thus ensuring that the learning game doesn't get boring. With the display format set to $\boxed{f} \boxed{FIX} \boxed{2}$, the execution of the program will cause the first problem to be displayed as follows: the display will show one number to the left of the decimal place, and one number to the right. For example, the numbers 8 and 2 would be displayed as 8.02. The user may then choose what operation to perform on the two numbers: he may add ($8 + 2$), subtract ($8 - 2$), multiply (8×2), or divide ($8 \div 2$). After he keys in his answer and re-initiates program execution, the program will either display a new problem, if his answer was right, or display the same two numbers again, but this time with a negative sign in front (-8.02). The negative sign is an indication that the answer was incorrect, and does not denote a negative number. (All numbers in the problems are positive, though of course the results of some subtractions may be negative). If the problem reappears with a negative sign, the user should key in a different answer and try again. As soon as the correct answer is given, the program will go to display a new problem.

DISPLAY		KEY ENTRY
LINE	CODE	
00		
01	24 01	RCL 1
02	15 73	$g\pi$
03	15 02	$g x^2$
04	61	x
05	15 01	$g \text{ FRAC}$
06	23 01	STO 1
07	24 00	RCL 0
08	61	x
09	14 01	f INT
10	23 03	STO 3
11	24 01	RCL 1
12	15 73	$g\pi$
13	15 02	$g x^2$
14	61	x
15	15 01	$g \text{ FRAC}$
16	23 01	STO 1
17	24 00	RCL 0
18	61	x
19	14 01	f INT
20	23 02	STO 2
21	24 03	RCL 3
22	33	EEX
23	02	2
24	71	\div

DISPLAY		KEY ENTRY
LINE	CODE	
25	51	+
26	23 04	STO 4
27	74	R/S
28	24 02	RCL 2
29	24 03	RCL 3
30	51	+
31	13 43	GTO 43
32	24 02	RCL 2
33	24 03	RCL 3
34	41	-
35	13 43	GTO 43
36	24 02	RCL 2
37	24 03	RCL 3
38	61	x
39	13 43	GTO 43
40	24 02	RCL 2
41	24 03	RCL 3
42	71	\div
43	14 71	f x=y
44	13 01	GTO 01
45	24 04	RCL 4
46	32	CHS
47	13 27	GTO 27
48		
49		

REGISTERS
R ₀ Max
R ₁ Random #
R ₂ Left #
R ₃ Right #
R ₄ Problem
R ₅
R ₆
R ₇

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Store Max ($0 < \text{Max} \leq 100$)	Max	STO	0			
3	Store seed ($0 < s < 1$)	s	STO	1			
4	Set display format		f	FIX	2		
5	Generate a problem		f	PRGM	R/S		n_1, n_2
6	Choose an operation and key in your answer:						
	For addition (+)	$n_1 + n_2$	R/S				
	For subtraction (-)	$n_1 - n_2$	GTO	32	R/S		
	For multiplication (x)	$n_1 \times n_2$	GTO	36	R/S		
	For division (\div)	$n_1 \div n_2$	GTO	40	R/S		
7	If you were right, program will display new problem; go to step 6.						n_3, n_4
8	If you were wrong, program will show same problem again; go to step 6 again.						$-n_1, n_2$
9	Repeat steps 6-8 as many times as desired						
10	To change Max, go to step 2, then to step 5.						

Example:

Let Max = 12 and the seed $s = 0.725$

Solution:

f PRGM R/S \longrightarrow 6.01
 (6 + 1 = 7)

7 R/S \longrightarrow 8.03
 (8 x 3 = 25)

25 GTO 3 6 R/S \longrightarrow -8.03
 (Try again: 8 x 3 = 24)

24 GTO 3 6 R/S \longrightarrow 3.11
 (3 - 11 = -8)

8 CHS GTO 3 2 R/S \longrightarrow 9.00
 (9 + 0 = 9)

9 R/S \longrightarrow 2.05

etc.

Notes

This image shows a single sheet of white paper with horizontal ruling lines. The lines are evenly spaced and run across the width of the page. There is no text or other markings on the paper.

CHAPTER 4 NAVIGATION

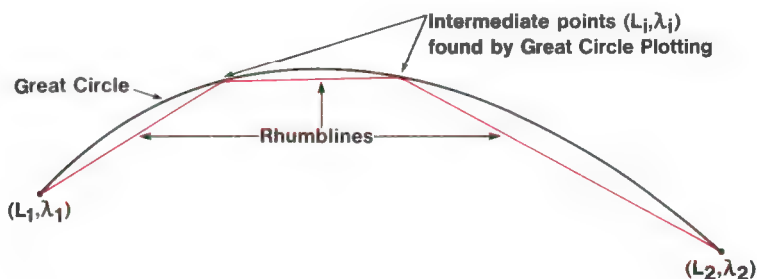
COURSE PLANNING—GREAT CIRCLE PLOTTING AND RHUMBLINE NAVIGATION

Long voyages by sea or air are generally made to follow one of two sorts of routes: a rhumbline or a great circle. The rhumbline is the path of constant heading between two points on the earth's surface; it intersects all lines of longitude at the same angle. It is also the course defined by the straight line between two points on a Mercator projection. It is a convenient course for navigation because its direction does not change, and for short distances at mid and low latitudes, the rhumbline is adequate for almost all calculations of course and distance.

Outside this range, a more efficient track is the great circle, which is always the shortest route between two points on a sphere. However, in order to follow a great circle, a vehicle must be continuously changing its course. Since this is at best inconvenient, if not impossible, several rhumblines are often used to approximate a great circle.

To plan a course using this technique, the navigator should first run the program Great Circle Plotting. For this program, the user must input the latitude and longitude of his starting point and his destination. Then, for any intermediate longitude λ_i that he specifies, the program will calculate the latitude L_i at which the great circle from source to destination will intersect the specified longitude. If several pairs of coordinates (L_i, λ_i) are calculated, then the next program, Rhumbline Navigation, may be used to find course and distance for the rhumblines linking these intermediate points along the great circle.

The inputs to Rhumbline Navigation are the coordinates of two points on the globe; outputs are the rhumbline course and distance from the first point to the second point. The program may be used alone, to determine the rhumbline from source to destination; or in conjunction with Great Circle Plotting, to compute several rhumblines to approximate a great circle.



GREAT CIRCLE PLOTTING

Equations:

$$L_i = \tan^{-1} \left[\frac{\tan L_2 \sin (\lambda_i - \lambda_1) - \tan L_1 \sin (\lambda_i - \lambda_2)}{\sin (\lambda_2 - \lambda_1)} \right]$$

where (L_1, λ_1) = coordinates of starting point (L_2, λ_2) = coordinates of destination (L_i, λ_i) = coordinates of intermediate point on great circle

Note:

The program does not compute along lines of longitude ($\lambda_1 = \lambda_2$).

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Input coordinates of starting point:						
	Latitude (CHS for S)	L_1 , D.MS	g	→H	STO	0	L_1 , dec. deg.
	Longitude (CHS for E)	λ_1 , D.MS	g	→H	STO	1	λ_1 , dec. deg.
3	Input coordinates of destination:						
	Latitude (CHS for S)	L_2 , D.MS	g	→H	STO	2	L_2 , dec. deg.
	Longitude (CHS for E)	λ_2 , D.MS	g	→H	STO	3	λ_2 , dec. deg.
4	Return to top of memory		f	PRGM			
5	Input the intermediate longitude (CHS for S) and compute cor- responding latitude	λ_i , D.MS	R/S				L_i , D.MS
6	For new intermediate longitude, go to step 5; for new source (or destination) go to step 2 (or step 3).						

RHUMBLINE NAVIGATION

Equations:

$$C = \tan^{-1} \frac{\pi (\lambda_1 - \lambda_2)}{180 \left[\ln \tan \left(45 + \frac{1}{2} L_2 \right) - \ln \tan \left(45 + \frac{1}{2} L_1 \right) \right]}$$

$$D = \begin{cases} 60 (\lambda_2 - \lambda_1) \cos L; \cos C = 0 \\ 60 \frac{(L_2 - L_1)}{\cos C}; \text{ otherwise} \end{cases}$$

where (L_1, λ_1) = coordinates of initial point (L_2, λ_2) = coordinates of final point

C = rhumbline course

D = rhumbline distance

Notes:

1. No course should pass through either the south or north pole.
2. The course may not go due east or due west across the 180° meridian (International Date Line).
3. Errors in distance calculations may be encountered as C approaches 90° or 270° .
4. Accuracy deteriorates for very short legs.

DISPLAY		KEY ENTRY	X	Y	Z	T	COMMENTS	REGISTERS
LINE	CODE							
00			λ_2	λ_1				R 0 L_1
01	41	-	$\lambda_1 - \lambda_2$					(dec. deg.)
02	23 06	STO 6	$\lambda_1 - \lambda_2$					
03	02	2		$\lambda_1 - \lambda_2$				R 1 λ_1
04	71	\div	α				Let $\alpha = 1/2 (\lambda_1 - \lambda_2)$	(dec. deg.)
05	14 04	f SIN	$\sin \alpha$				Normalize α so that	
06	15 04	g SIN ⁻¹	norm α				$-180 \leq \lambda_1 - \lambda_2 \leq 180$;	R 2 L_2
07	09	9	α				finds shortest route	(dec. deg.)
08	00	0	90	α			round earth	
09	71	\div	$\alpha/90$					R 3 λ_2
10	15 73	g π	π	$\alpha/90$				(dec. deg.)
11	61	x	$\pi\alpha/90$	$\pi\alpha/90$				
12	24 05	RCL 5	$\ln \tan_2$	$\pi\alpha/90$				R 4 $\ln \tan$
13	24 04	RCL 4	$\ln \tan_1$	y			Let $y = \pi\alpha/90$	(45+L ₁ /2)
14	41	-	x	y			Let $x = \ln \tan_2 - \ln \tan_1$	
15	15 09	g $\rightarrow P$	r	C			$C = \tan^{-1} y/x$	R 5 $\ln \tan$
16	22	R \downarrow	C			r		(45+L ₂ /2)
17	15 03	g ABS	C			r		
18	23 07	STO 7	C			r		R 6 $\lambda_1 - \lambda_2$
19	24 06	RCL 6	$\lambda_1 - \lambda_2$	C				
20	14 04	f SIN	$\sin 2\alpha$	C			Normalize $\lambda_1 - \lambda_2$ so	
21	15 04	g SIN ⁻¹	norm 2α	C			that $-90 \leq \lambda_1 - \lambda_2 \leq 90$	R 7 C
22	15 41	g x<0	2α	C			x < 0 means East to West	
23	13 26	GTO 26	2α	C				
24	21	x \leftrightarrow y	C	2α			W to E, C is answer	
25	13 31	GTO 31	C	2α				
26	03	3	3	2α	C		E to W, answer is	
27	06	6	36	2α	C		360 - C	
28	00	0	360	2α	C			
29	24 07	RCL 7	C	360	2α	C		
30	41	-	360 - C					
31	74	R/S	Course				Display course	
32	06	6	6				Compute distance D	
33	00	0	60					
34	24 07	RCL 7	C	60				
35	14 05	f COS	$\cos C $	60				
36	15 61	g x \neq 0	$\cos C $	60			If $\cos C \neq 0$,	
37	13 46	GTO 46	$\cos C $	60			go to line 45	
38	34	CLX	0	60			$\cos C = 0$; heading is	
39	24 06	RCL 6	$\lambda_1 - \lambda_2$				due E or due W	
40	61	x	60 ($\lambda_1 - \lambda_2$)					
41	24 02	RCL 2	L_2	60 ($\lambda_1 - \lambda_2$)				
42	14 05	f COS	$\cos L_2$	60 ($\lambda_1 - \lambda_2$)				
43	61	x	Dist				$D = 60 (\lambda_1 - \lambda_2) \cos L$	
44	13 00	GTO 00	Dist				Halt and display Dist	
45	71	\div	$60/\cos C $				Heading is not due E or W	
46	24 02	RCL 2	L_2				Apply formula:	
47	24 00	RCL 0	L_1	L_2	$60/\cos C $		$D = 60 (L_2 - L_1)/\cos C$	
48	41	-	$L_2 - L_1$	$60/\cos C $				
49	61	x	Dist				Halt	

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Input the initial latitude (CHS						
	for S)	L_1 , D.MS	g	→H	STO	2	
			2	÷	45	+	
			f	TAN	f	LN	
			STO	5			$\ln \tan_1$
3	Input the initial longitude (CHS						
	for E)	λ_1 , D.MS	g	→H	STO	3	λ_1 , dec. deg.
4	Input the final latitude (CHS for						
	S)	L_2 , D.MS	g	→H	RCL	2	
			STO	0	$x\bar{z}y$	STO	
			2	2	÷	45	
			+	f	TAN	f	
			LN	RCL	5	STO	
			4	$x\bar{z}y$	STO	5	$\ln \tan_2$
5	Input the final longitude (CHS						
	for E)	λ_2 , D.MS	g	→H	RCL	3	
			STO	1	$x\bar{z}y$	STO	
			3				λ_2 , dec. deg.
6	Compute course		f	PRGM	R/S		C
7	Compute distance		R/S				D
8	To continue the course, return to						
	step 4 and input a new final						
	position						

Example:

A ship sailing from San Francisco ($L37^\circ 49'N$, $\lambda 122^\circ 25'W$) to Tokyo ($L35^\circ 40'N$, $\lambda 139^\circ 45'E$) will follow three rhumb lines to approximate the great circle route. The navigator chooses the two intermediate points to be at $\lambda 155^\circ W$ and $\lambda 175^\circ E$. Find the rhumb line courses the ship should follow, and the distance covered on each leg.

Solution:

First key in Great Circle Plotting.

37.49 **g** **→H** **STO** **0** 122.25 **g** **→H** **STO** **1** 35.40 **g** **→H** **STO** **2** 139.45
CHS **g** **→H** **STO** **3** **f** **PRGM** 155 **R/S** → 47.4606
 175 **CHS** **R/S** → 47.3610

Thus the two intermediate points are ($L47^\circ 46'N$, $\lambda 155^\circ W$) and ($L47^\circ 36'N$, $\lambda 175^\circ E$).

DISPLAY		KEY ENTRY	X	Y	Z	T	COMMENTS	REGISTERS
LINE	CODE							
00								R ₀ Flag
01	34	CLX	0				Set flag to 0 for f(x)	
02	23 00	STO 0	0					
03	24 01	RCL 1	x	0			Recall x and branch to	R ₁ x
04	13 07	GTO 17	x	0			calculate f(x)	
05	22	R↓	f(x)				Roll down to remove flag	
06	23 04	STO 4	f(x)					R ₂ e
07	15 74	g NOP	f(x)				May Pause to see convergence	
08	01	1	1	f(x)			Set flag to 1 for f(x + δ)	
09	23 00	STO 0	1	f(x)				R ₃ δ
10	24 01	RCL 1	x	1	f(x)			
11	24 01	RCL 1	x	x	1	f(x)		
12	33	EEX	1. 00	x	x	1		R ₄ f(x)
13	05	5	1. 05	x	x	1		
14	71	÷	10 ⁻⁵ x	x	1	1		
15	23 03	STO 3	δ	x	1	1		R ₅
16	51	+	x + δ	1	1	1		
17							Lines 17 through 30 are	
18							reserved for user	R ₆
19							to define f(x)	
20								
21							This section of pgm is	R ₇
22							used to find f(x) and	
23							f(x + δ). Flag in R ₀ is	
24							0 for f(x), 1 for	
25							f(x + δ)	
26								
27								
28								
29								
30								
31	15 71	g x = 0	f(x)/(x + δ)				Is function value = 0?	
32	13 49	GTO 49	f(x)/(x + δ)				Yes, output solution	
33	24 00	RCL 0	Flag	f(x)/(x + δ)			No, check flag	
34	15 71	g x = 0	Flag	f(x)/(x + δ)			Flag = 0?	
35	13 05	GTO 05	Flag	f(x)			Yes, have f(x)	
36	22	R↓	f(x + δ)			Flag	No, flag = 1, have f(x + δ)	
37	24 04	RCL 4	f(x)	f(x + δ)				
38	71	÷	R				R = f(x + δ)/f(x)	
39	01	1	1	R				
40	41	-	R - 1				R - 1 = [f(x + δ) - f(x)]/f(x)	
41	15 03	g 1/x	(R - 1) ⁻¹				Approximate:	
42	24 03	RCL 3	δ	(R - 1) ⁻¹			f'(x) = [f(x + δ) - f(x)]/δ	
43	61	x	δ/(R - 1)				Δ = f(x)/f'(x)	
44	23 41 01	STO - 1	Δ				x _{i+1} = x _i - Δ	
45	15 03	g ABS	Δ					
46	24 02	RCL 2	e	Δ				
47	14 41	f x < y	e	Δ			x _{i+1} - x _i > ε?	
48	13 01	GTO 01	e	Δ			Yes, iterate again	
49	24 01	RCL 1	x	e	Δ		No, display x and halt	

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in lines 1-16 of program						16 51
2	Key in function $f(x)$						
3	Key in a branch to line 31		GTO	31			
4	Press SST until display shows line 30						
5	Key in lines 31-49 of program						
6	Switch to RUN						
7	Store initial guess for solution	x_1	STO	1			
8	Store tolerance	ϵ	STO	2			
9	Compute solution		f	PRGM	R/S		x_0
10	To change x_1 or ϵ go to appro- priate step and store new value.						

Example:

An equation often solved by gear designers is

$$\tan x - x - I = 0$$

where x is an angle in radians and I is the *involute* of x . Find the angle x_0 corresponding to an involute of 0.0324.

Note:

Since a gear designer might want to calculate x for several values of I , it will be simpler to store I in R_7 for use by the function $f(x)$.

Solution:*Example User Instructions*

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in lines 1-16 of program						16 51
2	Key in steps for $f(x) = \tan x -$ $x - 1$						
			f	TAN			17 14 06
			f	LASTx			18 14 73
			-				19 41
			RCL	7			20 24 07
			-				21 41
3	Key in branch to 31		GTO	31			22 13 31
4	Press SS1 8 times, until display shows line 30						
5	Key in lines 31-49						49 24 01
6	Switch to RUN						
7	Set angular mode		g	RAD			
8	Store I	.0324	STO	7			
9	Guess $x_1 = 1$	1	STO	1			
10	Set tolerance $\epsilon = 10^{-6}$	10^{-6}	STO	2			
11	Compute solution x_0		f	PRGM	R/S		0.45
12	Convert the angle to degrees		180	x	g	π	
			\div				25.62
13	Display last value of $f(x)$		RCL	4			2.30 -09

$$x_0 = 25.62^\circ$$

$$\text{Last } f(x) = 2.30 \times 10^{-9}$$

NUMERICAL INTEGRATION, SIMPSON'S RULE

Let x_0, x_1, \dots, x_n be equally spaced points such that $x_i = x_0 + ih$ for $i = 0, 1, 2, \dots, n$ at which corresponding values $f(x_0), f(x_1), \dots, f(x_n)$ of a function $f(x)$ are known. This function need not be known explicitly but if it is, these values can be found previously by writing the function into memory and evaluating at the various points. n must be an even positive integer.

Simpson's Rule is:

$$\int_{x_0}^{x_n} f(x) dx \cong \frac{h}{3} [f(x_0) + 4f(x_1) + 2f(x_2) + \dots + 4f(x_{n-3}) + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)].$$

Let the solution be indicated by I.

DISPLAY			KEY ENTRY	DISPLAY			KEY ENTRY	REGISTERS	
LINE	CODE			LINE	CODE				
00				25	61	x		R ₀ h/3	
01	24 00	RCL 0		26	24 01	RCL 1		R ₁ Σ	
02	03	3		27	51	+		R ₂	
03	71	÷		28	23 01	STO 1		R ₃	
04	23 00	STO 0		29	13 13	GTO 13		R ₄	
05	61	x		30				R ₅	
06	23 01	STO 1		31				R ₆	
07	74	R/S		32				R ₇	
08	24 00	RCL 0		33					
09	61	x		34					
10	24 01	RCL 1		35					
11	51	+		36					
12	23 01	STO 1		37					
13	74	R/S		38					
14	24 00	RCL 0		39					
15	61	x		40					
16	04	4		41					
17	61	x		42					
18	24 01	RCL 1		43					
19	51	+		44					
20	23 01	STO 1		45					
21	74	R/S		46					
22	24 00	RCL 0		47					
23	61	x		48					
24	02	2		49					

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Store increment	h	STO	0			
3	Enter first function value	$f(x_0)$	f	PRGM	R/S		Partial sum
4	Enter last function value	$f(x_n)$	R/S				Partial sum
5	Enter values $i = 1, 2, \dots, n - 2$	$f(x_i)$	R/S				Partial sum
6	Enter value $i = n - 1$	$f(x_{n-1})$	R/S				I

Example

Compute $\int_0^{\pi} \sin^2 x \, dx$ using Simpson's rule with $h = \pi/8$.

The following data must be found first:

i	0	1	2	3	4	5	6	7	8
x_i	0	$\pi/8$	$\pi/4$	$3\pi/8$	$\pi/2$	$5\pi/8$	$3\pi/4$	$7\pi/8$	π
$f(x_i)$	0	0.1464	0.5	0.8536	1	0.8536	0.5	0.1464	0

Solution:

$$\int_0^{\pi} \sin^2 x \, dx \cong 1.5708$$

The exact solution is $\pi/2$.

NUMERICAL SOLUTION TO DIFFERENTIAL EQUATIONS

This program may be used to solve a wide variety of first order differential equations of the form

$$y' = f(x, y)$$

with initial values x_0, y_0 .

The solution is a numerical solution which calculates y_i for $x_i = x_0 + ih$, where h is an increment specified by the user and $i = 1, 2, \dots$

The program uses a modified Euler method (predictor – corrector):

$$\hat{y}_{i+1} = y_i + h f(x_i, y_i) \quad y_{i+1} = y_i + \frac{h}{2} [f(x_i, y_i) + f(x_{i+1}, \hat{y}_{i+1})]$$

$f(x, y)$ is keyed into memory starting at line 18. The user has 13 program steps to write $f(x, y)$; registers R_5, R_6 , and R_7 are also available. The user should assume that x and y will be in the X- and Y-registers, respectively. The routine should return with the value of $f(x, y)$ in the X-register and should end with a GTO 31.

DISPLAY		KEY ENTRY
LINE	CODE	
00		
01	34	CLX
02	23 04	STO 4
03	24 02	RCL 2
04	24 01	RCL 1
05	13 18	GTO 18
06	22	R↓
07	23 03	STO 3
08	24 00	RCL 0
09	61	x
10	24 02	RCL 2
11	51	+
12	24 01	RCL 1
13	24 00	RCL 0
14	51	+
15	01	1
16	23 04	STO 4
17	22	R↓
18		
19		
20		
21		
22		
23		
24		

DISPLAY		KEY ENTRY
LINE	CODE	
25		
26		
27		
28		
29		
30		
31	24 04	RCL 4
32	15 71	g x=0
33	13 06	GTO 06
34	22	R↓
35	24 03	RCL 3
36	51	+
37	24 00	RCL 0
38	61	x
39	02	2
40	71	÷
41	24 02	RCL 2
42	51	+
43	23 02	STO 2
44	24 01	RCL 1
45	24 00	RCL 0
46	51	+
47	23 01	STO 1
48	14 74	f PAUSE
49	22	x↔y

REGISTERS	
R_0	h
R_1	x
R_2	y
R_3	y'
R_4	Flag
R_5	
R_6	
R_7	

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS	
1	Key in lines 1-17 of program						17	22
2	Key in function $f(x, y)$							
3	Key in branch to line 31		GTO	31				
4	Press SSR repeatedly until display shows line 30							
5	Key in lines 31-49 of program						49	13 01
6	Switch to RUN							
7	Store increment	h	STO	0				
8	Store initial conditions	x_0	STO	1				
		y_0	STO	2	f	PRGM		
9	Display next x-value and corresponding y-value							
			R/S					(x_k)
								y_k
10	Repeat step 9 as often as desired							

Example:

Solve numerically the differential equation $y' = x\sqrt{y}$ with initial conditions $x_0 = 1, y_0 = 1$. Use a step size of $h = 0.1$.

Solution:

Key the function in as **$x \times y$** **f** **\sqrt{x}** **\times**

x	1.0	1.1	1.2	1.3	1.4	1.5
y (by prgm)	1.0	1.1077	1.2319	1.3745	1.5372	1.7221
y (exact)	1.0	1.1078	1.2321	1.3748	1.5376	1.7227

LINEAR INTERPOLATION

If $(x_1, f(x_1))$ and $(x_2, f(x_2))$ are two points of a function $f(x)$, then the function at x_0 can be approximated by the following formula:

$$f(x_0) \cong \frac{(x_2 - x_0) f(x_1) + (x_0 - x_1) f(x_2)}{(x_2 - x_1)}$$

This is called the linear interpolation formula. Of course, x_2 cannot equal x_1 .

DISPLAY		KEY ENTRY
LINE	CODE	
00		
01	23 04	STO 4
02	24 00	RCL 0
03	41	-
04	24 03	RCL 3
05	61	x
06	24 02	RCL 2
07	24 04	RCL 4
08	41	-
09	24 01	RCL 1
10	61	x
11	51	+
12	24 02	RCL 2
13	24 00	RCL 0
14	41	-
15	71	÷
16	13 00	GTO 00
17		
18		
19		
20		
21		
22		
23		
24		

DISPLAY		KEY ENTRY
LINE	CODE	
25		
26		
27		
28		
29		
30		
31		
32		
33		
34		
35		
36		
37		
38		
39		
40		
41		
42		
43		
44		
45		
46		
47		
48		
49		

REGISTERS	
R ₀	x_1
R ₁	$f(x_1)$
R ₂	x_2
R ₃	$f(x_2)$
R ₄	x_0
R ₅	
R ₆	
R ₇	

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Store first point	x_1	STO	0			
		$f(x_1)$	STO	1			
3	Store second point	x_2	STO	2			
		$f(x_2)$	STO	3	f	PRGM	
4	Key in x_0 , find $f(x_0)$	x_0	R/S				$f(x_0)$
5	Repeat step 5 for as many x-						
	values as desired.						

Example:

Given

$$f(7.3) = 1.9879$$

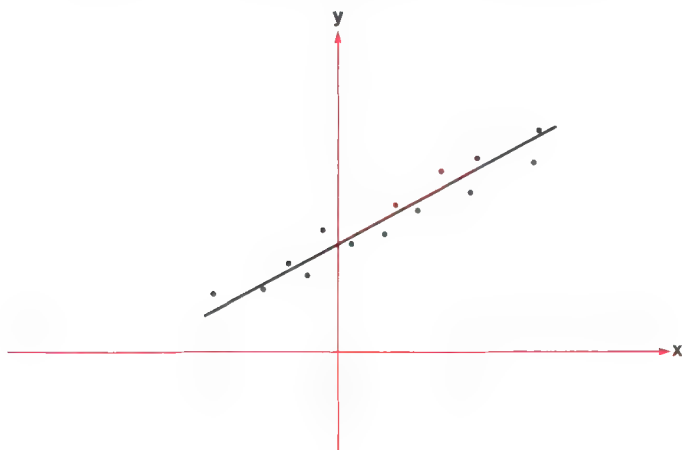
$$f(7.4) = 2.0015,$$

find by linear interpolation $f(7.37)$.**Solution:**

$$f(7.37) = 1.9974$$

CHAPTER 6 STATISTICS

CURVE FITTING—LINEAR REGRESSION



When investigating the relationship between two variables in the real world, it is a reasonable first step to make experimental observations of the system to gather paired values of the variables, (x, y) . The investigator might then ask the question: What mathematical formula best describes the relationship between the variables x and y ? His first guess will often be that the relationship is linear, i.e., that the form of the equation is $y = a_1 x + a_0$, where a_1 and a_0 are constants. The purpose of this program is to find the constants a_1 and a_0 , which give the closest agreement between the experimental data and the equation $y = a_1 x + a_0$. The technique used is linear regression by the method of least squares.

The user must input the paired values of data he has gathered, (x_i, y_i) , $i = 1, \dots, n$. When all data pairs have been input, the regression constants a_1 and a_0 may be calculated. A third value may also be found, the coefficient of determination, r^2 . The value of r^2 will lie between 0 and 1 and will indicate how closely the equation fits the experimental data: the closer r^2 is to 1, the better the fit.

Equations:

$$y = a_1 x + a_0$$

All summations below are performed for $i = 1, \dots, n$.

Regression constants:

$$a_1 = \frac{\Sigma xy - \frac{\Sigma x \Sigma y}{n}}{\Sigma x^2 - \frac{(\Sigma x)^2}{n}}$$

$$a_0 = \bar{y} - a_1 \bar{x}$$

where $\bar{y} = \frac{\Sigma y}{n}$

$$\bar{x} = \frac{\Sigma x}{n}$$

Coefficient of determination:

$$r^2 = \frac{\left[\Sigma xy - \frac{\Sigma x \Sigma y}{n} \right]^2}{\left[\Sigma x^2 - \frac{(\Sigma x)^2}{n} \right] \left[\Sigma y^2 - \frac{(\Sigma y)^2}{n} \right]}$$

Note:

The values for a_0 and a_1 are stored in R_0 and R_1 , respectively. After the calculation of a_0 , a_1 , and r^2 , the estimated y-value, \hat{y} , corresponding to any x-value may be calculated by $y = a_1 x + a_0$.

Programming Remarks:

The intermediate value $C = \Sigma xy - (\Sigma x \Sigma y / n)$ is first calculated at line 14 but is also needed near the end of the program to find r^2 . Since all registers R_0 through R_7 are in use, the only place to save this value is in the stack. Hence C is preserved in one or more of the stack registers from lines 14 through 36, when it is used. It is due to the presence of C in the stack that users are warned not to disturb the contents of the stack after calculation of a_0 and a_1 (see step 4 of User Instructions).

DISPLAY		KEY ENTRY	X	Y	Z	T	COMMENTS	REGISTERS
LINE	CODE							
00			y	x			Steps 1-7 for summation	R 0 a_0
01	31	f	y	y	x			
02	15 02	$g x^2$	y^2	y	x			
03	23 51 02	STO + 2	y^2	y	x		Σy^3	R 1 a_1
04	22	R↓	y	x		y^3		
05	21	$x \div y$	x	y		y^2		
06	25	$\Sigma +$	n	y		y^2	$n, \Sigma y, \Sigma xy, \Sigma x^2, \Sigma x$	R 2 Σy^2
07	13 00	GTO 00	n	y		y^2		
08	24 05	RCL 5	Σxy					
09	24 07	RCL 7	Σx	Σxy				R 3 n
10	24 04	RCL 4	Σy	Σx	Σxy			
11	61	x	$\Sigma x \Sigma y$	Σxy				
12	24 03	RCL 3	n	$\Sigma x \Sigma y$	Σxy			R 4 Σy
13	71	\div	$\Sigma x \Sigma y / n$	Σxy				
14	41	-	C				$C = \Sigma xy - (\Sigma x \Sigma y / n)$	
15	24 06	RCL 6	Σx^2	C				R 5 Σxy
16	24 07	RCL 7	Σx	Σx^2	C			
17	15 02	$g x^2$	$(\Sigma x)^2$	Σx^2	C			
18	24 03	RCL 3	n	$(\Sigma x)^2$	Σx^2	C		R 6 Σx^2
19	71	\div	$(\Sigma x)^2 / n$	Σx^2	C	C		
20	41	-	D	C	C	C	$D = \Sigma x^2 - ((\Sigma x)^2 / n)$	
21	71	\div	a_1	C	C	C	$a_1 = C/D$	R 7 Σx
22	23 01	STO 1	a_1	C	C	C		
23	24 07	RCL 7	Σx	a_1	C	C		
24	61	x	$a_1 \Sigma x$	C	C	C		
25	32	CHS	$-a_1 \Sigma x$	C	C	C		
26	24 04	RCL 4	Σy	$-a_1 \Sigma x$	C	C		
27	51	+	$\Sigma y - a_1 \Sigma x$	C	C	C		
28	24 03	RCL 3	n	$\Sigma y - a_1 \Sigma x$	C	C		
29	71	\div	a_0	C	C	C	$a_0 = \bar{y} - a_1 \bar{x}$	
30	23 00	STO 0	a_0	C	C	C		
31	74	R/S	a_0	C	C	C	Halt to display a_0	
32	24 01	RCL 1	a_1	a_0	C	C		
33	74	R/S	a_1	a_0	C	C	Halt to display a_1	
34	21	$x \div y$	a_0	a_1	C	C		
35	22	R↓	a_1	C	C	a_0		
36	61	x	$a_1 C$	C	a_0	a_0		
37	24 02	RCL 2	Σy^3	$a_1 C$	C	a_0		
38	24 04	RCL 4	Σy	Σy^2	$a_1 C$	C		
39	15 02	$g x^2$	$(\Sigma y)^2$	Σy^2	$a_1 C$	C		
40	24 03	RCL 3	n	$(\Sigma y)^2$	Σy^2	$a_1 C$		
41	71	\div	$(\Sigma y)^2 / n$	Σy^2	$a_1 C$	$a_1 C$		
42	41	-	E	$a_1 C$	$a_1 C$	$a_1 C$	$E = \Sigma y^2 - ((\Sigma y)^2 / n)$	
43	71	\div	r^2	$a_1 C$	$a_1 C$	$a_1 C$	$r^2 = a_1 C/E$	
44	13 00	GTO 00	r^2	$a_1 C$	$a_1 C$	$a_1 C$		
45								
46								
47								
48								
49								

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Initialize		f	REG	f	PRGM	
3	Perform for $i = 1, \dots, n$:						
	Input x-value and y-value	x_i	\uparrow				
		y_i	R/S				i
4	Compute regression constants		GTO	08	R/S		a_0^*
			R/S				a_1^*
5	Compute coefficient of determination						r^2
6	To calculate a projected y-value,						
	input the x-value	x	RCL	1	\times	RCL	
			0	+			\hat{y}
7	Perform step 6 as many times as						
	desired						
8	For a new case, go to step 2.						
	* The contents of the stack						
	should not be disturbed at these						
	points.						

Example:

An eccentric professor of numerical analysis wakes up one morning and feels feverish. A search through his medicine cabinet reveals one oral thermometer which, unfortunately, is in degrees centigrade, a scale he is not familiar with. As he stares disconsolately out his window, he spies the outdoor thermometer affixed to the windowframe. This thermometer, however, will not fit comfortably into his mouth. Still, with some ingenuity....

The professor suspects that the relationship is $F = a_1 C + a_0$. If he can get a few data pairs for F and C , he can run a linear regression program to find a_1 and a_0 , then convert any reading in $^{\circ}\text{C}$ to $^{\circ}\text{F}$ through the equation. So tossing both thermometers into a sink of lukewarm water, he reads the following pairs of temperatures as the water cools:

C	40.5	38.6	37.9	36.2	35.1	34.6
F	104.5	102	100	97.5	95.5	94

If the relationship is indeed $F = a_1 C + a_0$, what are the values for a_1 and a_0 ? What is the coefficient of determination?

Solution:

[f]	[PRGM]	[f]	[REG]	40.5	[+]	104.5	[R/S]	→	1.00	
38.6	[+]	102	[R/S]						→	2.00
37.9	[+]	100	[R/S]						→	3.00
36.2	[+]	97.5	[R/S]						→	4.00
35.1	[+]	95.5	[R/S]						→	5.00
34.6	[+]	94	[R/S]						→	6.00
[GTO]	[0]	[8]	[R/S]						→	33.53
[R/S]									→	1.76
[R/S]									→	0.99

Thus, by the data above, $F = 1.76 C + 33.53$, with $r^2 = 0.99$. (The real equation, of course, is $F = 1.8C + 32$.)

Suppose the professor puts the centigrade thermometer in his mouth and finds he has a temperature of 37°C . Should he be worried?

37 [RCL] [1] [x] [RCL] [0] [+] → 98.65°F

It looks like he is safe.

EXPONENTIAL CURVE FIT

This program computes the least squares fit of n pairs of data points $\{(x_i, y_i), i = 1, 2, \dots, n\}$, where $y_i > 0$, for an exponential function of the form

$$y = a e^{bx} \quad (a > 0).$$

The equation is linearized into

$$\ln y = \ln a + bx.$$

The following statistics are computed:

1. Coefficients a, b

$$b = \frac{\sum x_i \ln y_i - \frac{1}{n} (\sum x_i)(\sum \ln y_i)}{\sum x_i^2 - \frac{1}{n} (\sum x_i)^2}$$

$$a = \exp \left[\frac{\sum \ln y_i}{n} - b \frac{\sum x_i}{n} \right]$$

2. Coefficient of determination

$$r^2 = \frac{\left[\sum x_i \ln y_i - \frac{1}{n} \sum x_i \sum \ln y_i \right]^2}{\left[\sum x_i^2 - \frac{(\sum x_i)^2}{n} \right] \left[\sum (\ln y_i)^2 - \frac{(\sum \ln y_i)^2}{n} \right]}$$

3. Estimated value \hat{y} for a given x

$$\hat{y} = a e^{bx}$$

Note:

n is a positive integer and $n \neq 1$.

DISPLAY		KEY ENTRY
LINE	CODE	
00		
01	14 07	f LN
02	31	↑
03	15 02	$g x^2$
04	23 51 02	STO + 2
05	22	R↓
06	21	$x \div y$
07	25	$\Sigma +$
08	13 00	GTO 00
09	24 05	RCL 5
10	24 07	RCL 7
11	24 04	RCL 4
12	61	x
13	24 03	RCL 3
14	71	÷
15	41	-
16	24 06	RCL 6
17	24 07	RCL 7
18	15 02	$g x^2$
19	24 03	RCL 3
20	71	÷
21	41	-
22	71	÷
23	23 01	STO 1
24	24 07	RCL 7

DISPLAY		KEY ENTRY
LINE	CODE	
25	61	x
26	32	CHS
27	24 04	RCL 4
28	51	+
29	24 03	RCL 3
30	71	÷
31	15 07	$g e^x$
32	23 00	STO 0
33	74	R/S
34	24 01	RCL 1
35	74	R/S
36	21	$x \div y$
37	22	R↓
38	61	x
39	24 02	RCL 2
40	24 04	RCL 4
41	15 02	$g x^2$
42	24 03	RCL 3
43	71	÷
44	41	-
45	71	÷
46	13 00	GTO 00
47		
48		
49		

REGISTERS	
R ₀	a
R ₁	b
R ₂	$\Sigma (\ln y)^2$
R ₃	n
R ₄	$\Sigma \ln y$
R ₅	$\Sigma x \ln y$
R ₆	Σx^2
R ₇	Σx

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Initialize		f	REG	f	PRGM	
3	Perform for $i = 1, \dots, n$:						
	Input x-value and y-value	x_i	↑				
		y_i	R/S				i
4	Compute constants		GTO	09	R/S		a^*
			R/S				b^*
5	Compute coefficient of determination						
			R/S				r^2
6	To calculate \hat{y} , input x	x	RCL	1	x	g	
			e^x	RCL	0	x	\hat{y}
7	Perform step 6 as many times as desired						
8	For new case, go to step 2.						
	* The stack must be maintained at these points.						

Example:

x_i	.72	1.31	1.95	2.58	3.14
y_i	2.16	1.61	1.16	.85	0.5

Solution:

$$a = 3.45, b = -0.58$$

$$y = 3.45 e^{-0.58x}$$

$$r^2 = 0.98$$

$$\text{For } x = 1.5, \hat{y} = 1.44$$

LOGARITHMIC CURVE FIT

This program fits a logarithmic curve

$$y = a + b \ln x$$

to a set of data points

$$\{(x_i, y_i), i = 1, 2, \dots, n\}$$

where $x_i > 0$.

Program computes:

1. Regression coefficients

$$b = \frac{\sum y_i \ln x_i - \frac{1}{n} \sum \ln x_i \sum y_i}{\sum (\ln x_i)^2 - \frac{1}{n} (\sum \ln x_i)^2}$$

$$a = \frac{1}{n} (\sum y_i - b \sum \ln x_i)$$

2. Coefficient of determination

$$r^2 = \frac{\left[\sum y_i \ln x_i - \frac{1}{n} \sum \ln x_i \sum y_i \right]^2}{\left[\sum (\ln x_i)^2 - \frac{1}{n} (\sum \ln x_i)^2 \right] \left[\sum y_i^2 - \frac{1}{n} (\sum y_i)^2 \right]}$$

3. Estimated value \hat{y} for given x

$$\hat{y} = a + b \ln x$$

Note:

n is a positive integer and $n \neq 1$.

DISPLAY		KEY ENTRY
LINE	CODE	
00		
01	31	\uparrow
02	15 02	$g x^2$
03	23 51 02	STO + 2
04	22	$R\downarrow$
05	21	$x\leftrightarrow y$
06	14 07	f LN
07	25	$\Sigma+$
08	13 00	GTO 00
09	24 05	RCL 5
10	24 07	RCL 7
11	24 04	RCL 4
12	61	\times
13	24 03	RCL 3
14	71	\div
15	41	$-$
16	24 06	RCL 6
17	24 07	RCL 7
18	15 02	$g x^2$
19	24 03	RCL 3
20	71	\div
21	41	$-$
22	71	\div
23	23 01	STO 1
24	24 07	RCL 7

DISPLAY		KEY ENTRY
LINE	CODE	
25	61	\times
26	32	CHS
27	24 04	RCL 4
28	51	$+$
29	24 03	RCL 3
30	71	\div
31	23 00	STO 0
32	74	R/S
33	24 01	RCL 1
34	74	R/S
35	21	$x\leftrightarrow y$
36	22	$R\downarrow$
37	61	\times
38	24 02	RCL 2
39	24 04	RCL 4
40	15 02	$g x^2$
41	24 03	RCL 3
42	71	\div
43	41	$-$
44	71	\div
45	13 00	GTO 00
46		
47		
48		
49		

REGISTERS	
R_0	a
R_1	b
R_2	Σy^2
R_3	n
R_4	Σy
R_5	$\Sigma y \ln x$
R_6	$\Sigma \ln x$
R_7	$\Sigma (\ln x)^2$

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Initialize		f	REG	f	PRGM	
3	Perform for $i = 1, \dots, n$:						
	Input x-value and y-value	x_i	\uparrow				
		y_i	R/S				i
4	Compute constants		GTO	09	R/S		a^*
			R/S				b^*
5	Compute coefficient of determination						r^2
6	To calculate \hat{y} , input x	x	f	ln	RCL	1	
			x	RCL	0	+	\hat{y}
7	Perform step 6 as many times as desired						
8	For new case, go to step 2.						
	* The stack must be maintained at these points						

Example:

x_i	3	4	6	10	12
y_i	1.5	9.3	23.4	45.8	60.1

Solution:

$$a = -47.02, b = 41.39$$

$$y = -47.02 + 41.39 \ln x$$

$$r^2 = 0.98$$

$$\text{For } x = 8, \hat{y} = 39.06$$

$$\text{For } x = 14.5, \hat{y} = 63.67$$

POWER CURVE FIT

This program fits a power curve

$$y = ax^b \quad (a > 0)$$

to a set of data points

$$\{(x_i, y_i), i = 1, 2, \dots, n\}$$

where $x_i > 0, y_i > 0$.

By writing this equation as

$$\ln y = b \ln x + \ln a$$

the problem can be solved as a linear regression problem.

Output statistics are:

1. Regression coefficients

$$b = \frac{\sum (\ln x_i) (\ln y_i) - \frac{(\sum \ln x_i) (\sum \ln y_i)}{n}}{\sum (\ln x_i)^2 - \frac{(\sum \ln x_i)^2}{n}}$$

$$a = \exp \left[\frac{\sum \ln y_i}{n} - b \frac{\sum \ln x_i}{n} \right]$$

2. Coefficient of determination

$$r^2 = \frac{\left[\sum (\ln x_i) (\ln y_i) - \frac{(\sum \ln x_i) (\sum \ln y_i)}{n} \right]^2}{\left[\sum (\ln x_i)^2 - \frac{(\sum \ln x_i)^2}{n} \right] \left[\sum (\ln y_i)^2 - \frac{(\sum \ln y_i)^2}{n} \right]}$$

3. Estimated value \hat{y} for given x

$$\hat{y} = ax^b$$

Note:

n is a positive integer and $n \neq 1$.

DISPLAY		KEY ENTRY
LINE	CODE	
00		
01	14 07	f LN
02	31	↑
03	15 02	g x ²
04	23 51 02	STO + 2
05	22	R↓
06	21	x↔y
07	14 07	f LN
08	25	Σ+
09	13 00	GTO 00
10	24 05	RCL 5
11	24 07	RCL 7
12	24 04	RCL 4
13	61	x
14	24 03	RCL 3
15	71	÷
16	41	-
17	24 06	RCL 6
18	24 07	RCL 7
19	15 02	g x ²
20	24 03	RCL 3
21	71	÷
22	41	-
23	71	÷
24	23 01	STO 1

DISPLAY		KEY ENTRY
LINE	CODE	
25	24 07	RCL 7
26	61	x
27	32	CHS
28	24 04	RCL 4
29	51	+
30	24 03	RCL 3
31	71	÷
32	15 07	g e ^x
33	23 00	STO 0
34	74	R/S
35	24 01	RCL 1
36	74	R/S
37	21	x↔y
38	22	R↓
39	61	x
40	24 02	RCL 2
41	24 04	RCL 4
42	15 02	g x ²
43	24 03	RCL 3
44	71	÷
45	41	-
46	71	÷
47	13 00	GTO 00
48		
49		

REGISTERS	
R ₀	a
R ₁	b
R ₂	Σ (ln y) ²
R ₃	n
R ₄	Σ ln y
R ₅	Σ (ln x) (ln y)
R ₆	Σ (ln x) ²
R ₇	Σ ln x

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Initialize		f	REG	f	PRGM	
3	Perform for i = 1, ..., n:						
	Input x-value and y-value	x _i	↑				
		y _i	R/S				i
4	Compute constants		GTO	10	R/S		a*
			R/S				b*
5	Compute coefficient of determination		R/S				r ²
6	Input x-value and compute \hat{y}	x	RCL	1	f	y ^x	
			RCL	0	x		\hat{y}
7	Perform step 6 as many times as desired						
8	For new case, go to step 2.						
	* The stack must be maintained at these points.						

Example:

x_i	10	12	15	17	20	22	25	27	30	32	35
y_i	0.95	1.05	1.25	1.41	1.73	2.00	2.53	2.98	3.85	4.59	6.02

Solution:

$$a = .03, b = 1.46$$

$$y = .03x^{1.46}$$

$$r^2 = 0.94$$

$$\text{For } x = 18, \hat{y} = 1.76$$

$$x = 23, \hat{y} = 2.52$$

COVARIANCE AND CORRELATION COEFFICIENT

For a set of given data points $\{(x_i, y_i), i = 1, 2, \dots, n\}$, the covariance and the correlation coefficient are defined as:

$$\text{covariance } s_{xy} = \frac{1}{n-1} \left(\sum x_i y_i - \frac{1}{n} \sum x_i \sum y_i \right)$$

$$\text{or } s_{xy}' = \frac{1}{n} \left(\sum x_i y_i - \frac{1}{n} \sum x_i \sum y_i \right)$$

$$\text{correlation coefficient } r = \frac{s_{xy}}{s_x s_y}$$

where s_x and s_y are standard deviations

$$s_x = \sqrt{\frac{\sum x_i^2 - (\sum x_i)^2/n}{n-1}} \quad s_y = \sqrt{\frac{\sum y_i^2 - (\sum y_i)^2/n}{n-1}}$$

Note:

$$-1 \leq r \leq 1$$

DISPLAY		KEY ENTRY
LINE	CODE	
00		
01	31	↑
02	15 02	$g x^2$
03	23 51 02	STO + 2
04	22	R↓
05	21	xz^2y
06	25	$\Sigma +$
07	13 00	GTO 00
08	24 05	RCL 5
09	24 04	RCL 4
10	24 07	RCL 7
11	61	x
12	24 03	RCL 3
13	71	÷
14	41	-
15	24 03	RCL 3
16	01	1
17	41	-
18	23 00	STO 0
19	71	÷
20	23 01	STO 1
21	74	R/S
22	24 00	RCL 0
23	61	x
24	24 03	RCL 3

DISPLAY		KEY ENTRY
LINE	CODE	
25	71	÷
26	74	R/S
27	14 22	f s
28	23 71 01	STO ÷ 1
29	24 02	RCL 2
30	24 04	RCL 4
31	15 02	$g x^2$
32	24 03	RCL 3
33	71	÷
34	41	-
35	24 00	RCL 0
36	71	÷
37	14 02	$f \sqrt{x}$
38	23 71 01	STO ÷ 1
39	24 01	RCL 1
40	13 00	GTO 00
41		
42		
43		
44		
45		
46		
47		
48		
49		

REGISTERS
R ₀ n - 1
R ₁ Used
R ₂ Σy^2
R ₃ n
R ₄ Σy
R ₅ Σxy
R ₆ Σx^2
R ₇ Σx

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Initialize		f	PRGM	f	REG	
3	Perform this step for $i = 1, 2, \dots, n$	x_i	\uparrow				
		y_i	R/S				i
4	Compute covariance s_{xy}		GTO	08	R/S		s_{xy}
5	Compute s_{xy}'		R/S				s_{xy}'
6	Compute correlation coefficient		R/S				r
7	For new case, go to step 2.						

Example:

x_i	26	30	44	50	62	68	74
y_i	92	85	78	81	54	51	40

Solution:

$$s_{xy} = -354.14$$

$$s_{xy}' = -303.55$$

$$r = -0.96$$

MOMENTS AND SKEWNESS

This program computes the following statistics for a set of given data $\{x_1, x_2, \dots, x_n\}$:

$$1^{\text{st}} \text{ moment } \quad \bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$$

$$2^{\text{nd}} \text{ moment } \quad m_2 = \frac{1}{n} \sum x_i^2 - \bar{x}^2$$

$$3^{\text{rd}} \text{ moment } \quad m_3 = \frac{1}{n} \sum x_i^3 - \frac{3}{n} \bar{x} \sum x_i^2 + 2\bar{x}^3$$

moment coefficient of skewness

$$\gamma_1 = \frac{m_3}{m_2^{3/2}}$$

DISPLAY		KEY ENTRY
LINE	CODE	
00		
01	31	↑
02	15 02	g x ²
03	25	Σ+
04	13 00	GTO 00
05	24 04	RCL 4
06	24 03	RCL 3
07	71	÷
08	23 02	STO 2
09	74	R/S
10	24 07	RCL 7
11	24 03	RCL 3
12	71	÷
13	24 02	RCL 2
14	15 02	g x ²
15	41	-
16	23 01	STO 1
17	74	R/S
18	24 05	RCL 5
19	24 03	RCL 3
20	71	÷
21	24 07	RCL 7
22	24 02	RCL 2
23	61	x
24	24 03	RCL 3

DISPLAY		KEY ENTRY
LINE	CODE	
25	71	÷
26	03	3
27	61	x
28	41	-
29	24 02	RCL 2
30	31	↑
31	15 02	g x ²
32	61	x
33	02	2
34	61	x
35	51	+
36	23 00	STO 0
37	74	R/S
38	24 00	RCL 0
39	24 01	RCL 1
40	01	1
41	73	·
42	05	5
43	14 03	f y ^x
44	71	÷
45	13 00	GTO 00
46		
47		
48		
49		

REGISTERS	
R ₀	m ₃
R ₁	m ₂
R ₂	\bar{x}
R ₃	n
R ₄	Σ x
R ₅	Σ x ³
R ₆	Σ x ⁴
R ₇	Σ x ²

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Initialize		f	PRGM	f	REG	
3	Perform for $i = 1, 2, \dots, n$:						
	Input x-value	x_i	R/S				i
4	Delete erroneous data	x_k	\uparrow	g	x^2	f	
			$\Sigma-$				
5	Compute the mean		GTO	05	R/S		\bar{x}
6	Compute the second and third moments						
			R/S				m_2
			R/S				m_3
7	Compute the moment coefficient of skewness						
			R/S				γ_1
8	For new case, go to step 2.						

Example:

i	1	2	3	4	5	6	7	8	9
x_i	2.1	3.5	4.2	6.5	4.1	3.6	5.3	3.7	4.9

Solution:

$$\bar{x} = 4.21$$

$$m_2 = 1.39$$

$$m_3 = 0.39$$

$$\gamma_1 = 0.24$$

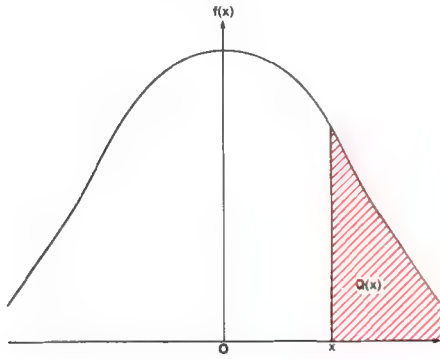
NORMAL DISTRIBUTION

The density function for a standard normal variable is

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

The upper tail area is

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-\frac{t^2}{2}} dt.$$



For $x \geq 0$, polynomial approximation is used to compute $Q(x)$:

$$Q(x) = f(x) (b_1 t + b_2 t^2 + b_3 t^3 + b_4 t^4 + b_5 t^5) + \epsilon(x)$$

where $|\epsilon(x)| < 7.5 \times 10^{-8}$

$$t = \frac{1}{1 + rx}, \quad r = 0.2316419$$

$$b_1 = .31938153, \quad b_2 = -.356563782$$

$$b_3 = 1.781477937, \quad b_4 = -1.821255978$$

$$b_5 = 1.330274429$$

Note:

The program only works for $x \geq 0$. Equations $f(-x) = f(x)$, $Q(-x) = 1 - Q(x)$, where $x \geq 0$, can be used to find f and Q for negative numbers.

Reference:

Abramowitz and Stegun, *Handbook of Mathematical Functions*, National Bureau of Standards, 1968.

DISPLAY		KEY ENTRY
LINE	CODE	
00		
01	31	\uparrow
02	23 06	STO 6
03	61	x
04	02	2
05	71	\div
06	32	CHS
07	15 07	$g e^x$
08	15 73	$g \pi$
09	02	2
10	61	x
11	14 02	$f \sqrt{x}$
12	71	\div
13	23 07	STO 7
14	74	R/S
15	24 00	RCL 0
16	24 06	RCL 6
17	61	x
18	01	1
19	51	+
20	15 22	$g 1/x$
21	31	\uparrow
22	31	\uparrow
23	31	\uparrow
24	24 05	RCL 5

DISPLAY		KEY ENTRY
LINE	CODE	
25	61	x
26	24 04	RCL 4
27	51	+
28	61	x
29	24 03	RCL 3
30	51	+
31	61	x
32	24 02	RCL 2
33	51	+
34	61	x
35	24 01	RCL 1
36	51	+
37	61	x
38	24 07	RCL 7
39	61	x
40	13 00	GTO 00
41		
42		
43		
44		
45		
46		
47		
48		
49		

REGISTERS	
R ₀	r
R ₁	b ₁
R ₂	b ₂
R ₃	b ₃
R ₄	b ₄
R ₅	b ₅
R ₆	x
R ₇	f(x)

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Initialize		f	PRGM			
3	Store constants	r	STO	0			
		b ₁	STO	1			
		b ₂	STO	2			
		b ₃	STO	3			
		b ₄	STO	4			
		b ₅	STO	5			
4	Input x and compute f(x)	x	R/S				f(x)
5	Compute Q(x)		R/S				Q(x)
6	For a new case, go to 4.						

Examples:

1. $x = 1.18$

2. $x = 2.28$

Solutions:

1. $f(x) = 0.20$

$Q(x) = 0.12$

2. $f(x) = 0.03$

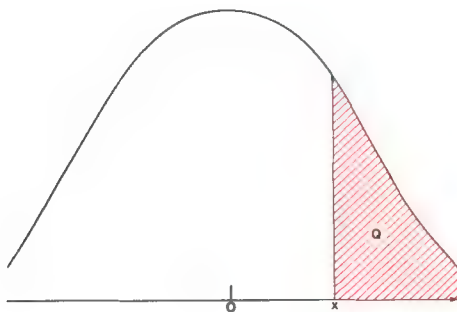
$Q(x) = 0.01$

INVERSE NORMAL INTEGRAL

This program determines the value of x such that

$$Q = \int_x^{\infty} \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}} dt$$

where Q is given and $0 < Q \leq 0.5$.



The following rational approximation is used:

$$x = t - \frac{c_0 + c_1 t + c_2 t^2}{1 + d_1 t + d_2 t^2 + d_3 t^3} + \epsilon(Q)$$

where $|\epsilon(Q)| < 4.5 \times 10^{-4}$

$$t = \sqrt{\ln \frac{1}{Q^2}}$$

$$c_0 = 2.515517 \quad d_1 = 1.432788$$

$$c_1 = 0.802853 \quad d_2 = 0.189269$$

$$c_2 = 0.010328 \quad d_3 = 0.001308$$

Reference:

Abramowitz and Stegun, *Handbook of Mathematical Functions*, National Bureau of Standards, 1968.

DISPLAY		KEY ENTRY
LINE	CODE	
00		
01	31	↑
02	61	x
03	15 22	g 1/x
04	14 07	f LN
05	14 02	f √x
06	23 06	STO 6
07	31	↑
08	31	↑
09	31	↑
10	24 05	RCL 5
11	61	x
12	24 04	RCL 4
13	51	+
14	61	x
15	24 03	RCL 3
16	51	+
17	61	x
18	01	1
19	51	+
20	23 07	STO 7
21	34	CLX
22	24 02	RCL 2
23	61	x
24	24 01	RCL 1

DISPLAY		KEY ENTRY
LINE	CODE	
25	51	+
26	61	x
27	24 00	RCL 0
28	51	+
29	24 07	RCL 7
30	71	÷
31	41	-
32	13 00	GTO 00
33		
34		
35		
36		
37		
38		
39		
40		
41		
42		
43		
44		
45		
46		
47		
48		
49		

REGISTERS
R ₀ C ₀
R ₁ C ₁
R ₂ C ₂
R ₃ d ₁
R ₄ d ₂
R ₅ d ₃
R ₆ t
R ₇ 1 + d ₁ t + d ₂ t ² + d ₃ t ³

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Initialize		f	PRGM			
3	Store constants	C ₀	STO	0			
		C ₁	STO	1			
		C ₂	STO	2			
		d ₁	STO	3			
		d ₂	STO	4			
		d ₃	STO	5			
4	Input Q	Q	R/S				
5	For a new case, go to 4.						

Examples:

1. $Q = 0.12$
2. $Q = 0.05$

Solutions:

1. $x = 1.18$
2. $x = 1.65$

FACTORIAL

This program will compute factorials for positive integers between 2 and 69.

$$n! = n(n-1)(n-2) \dots (2)(1)$$

Notes:

- For large values of n , the program will take some time to arrive at a result, up to a maximum of about 20 seconds for $n = 69$.
- The program does not check input values and will return incorrect answers for values of $n < 2$ or $n > 69$ or n non-integer.

DISPLAY		KEY ENTRY
LINE	CODE	
00		
01	01	1
02	23 00	STO 0
03	21	$x \leftrightarrow y$
04	23 61 00	STO x 0
05	01	1
06	41	-
07	14 61	$f x \neq y$
08	13 04	GTO 04
09	24 00	RCL 0
10	13 00	GTO 00
11		
12		
13		
14		
15		
16		
17		
18		
19		
20		
21		
22		
23		
24		

DISPLAY		KEY ENTRY
LINE	CODE	
25		
26		
27		
28		
29		
30		
31		
32		
33		
34		
35		
36		
37		
38		
39		
40		
41		
42		
43		
44		
45		
46		
47		
48		
49		

REGISTERS	
R ₀	Used
R ₁	
R ₂	
R ₃	
R ₄	
R ₅	
R ₆	
R ₇	

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Initialize		f	PRGM			
3	Key in n ($2 \leq n \leq 69$)	n	R/S				n!
4	For a new n, go to step 3.						

Examples:

1. $5! = 120.00$
2. $10! = 3628800.00$

PERMUTATION

A permutation is an ordered subset of a set of distinct objects. The number of possible permutations, each containing n objects, that can be formed from a collection of m distinct objects is given by

$${}_m P_n = \frac{m!}{(m-n)!} = m(m-1) \dots (m-n+1)$$

where m, n are integers and $0 \leq n \leq m$.

Notes:

1. ${}_m P_n$ can also be denoted by P_n^m , $P(m, n)$ or $(m)_n$.
2. ${}_m P_0 = 1$, ${}_m P_1 = m$, ${}_m P_m = m!$

DISPLAY		KEY ENTRY
LINE	CODE	
00		
01	24 00	RCL 0
02	24 00	RCL 0
03	24 01	RCL 1
04	15 71	g x=0
05	13 29	GTO 29
06	14 71	f x=y
07	13 31	GTO 31
08	14 51	f x>y
09	13 39	GTO 39
10	01	1
11	14 71	f x=y
12	13 41	GTO 41
13	22	R↓
14	41	-
15	01	1
16	51	+
17	61	x
18	14 73	f LASTx
19	24 00	RCL 0
20	01	1
21	41	-
22	14 71	f x=y
23	13 26	GTO 26
24	22	R↓

DISPLAY		KEY ENTRY
LINE	CODE	
25	13 15	GTO 15
26	22	R↓
27	22	R↓
28	13 00	GTO 00
29	01	1
30	13 00	GTO 00
31	01	1
32	41	-
33	15 71	g x=0
34	13 37	GTO 37
35	23 61 00	STO x 0
36	13 31	GTO 31
37	24 00	RCL 0
38	13 00	GTO 00
39	00	0
40	71	÷
41	22	R↓
42	22	R↓
43	13 00	GTO 00
44		
45		
46		
47		
48		
49		

REGISTERS	
R ₀ m	
R ₁ n	
R ₂	
R ₃	
R ₄	
R ₅	
R ₆	
R ₇	

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Store m, n	m	STO	0			
		n	STO	1			
3	Compute permutations		f	PRGM	R/S		${}_m P_n$
4	For new case, go to step 2.						

Examples:

- ${}_{43}P_3 = 74046.00$
- ${}_{73}P_4 = 26122320.00$

COMBINATION

A combination is a selection of one or more of a set of distinct objects without regard to order. The number of possible combinations, each containing n objects, that can be formed from a collection of m distinct objects is given by

$${}_m C_n = \frac{m!}{(m-n)! n!} = \frac{m(m-1) \dots (m-n+1)}{1 \cdot 2 \cdot \dots \cdot n}$$

where m, n are integers and $0 \leq n \leq m$.

This program computes ${}_m C_n$ using the following algorithm:

1. If $n \leq m - n$

$${}_m C_n = \frac{m-n+1}{1} \cdot \frac{m-n+2}{2} \cdot \dots \cdot \frac{m}{n}.$$

2. If $n > m - n$, program computes ${}_m C_{m-n}$.

Notes:

1. ${}_m C_n$, which is also called the binomial coefficient, can be denoted by C_n^m , $C(m,n)$, or $\binom{m}{n}$.
2. ${}_m C_n = {}_m C_{m-n}$
3. ${}_m C_0 = {}_m C_m = 1$
4. ${}_m C_1 = {}_m C_{m-1} = m$

DISPLAY		KEY ENTRY
LINE	CODE	
00		
01	41	-
02	14 73	f LASTx
03	14 41	f x<y
04	21	x \leftrightarrow y
05	23 00	STO 0
06	01	1
07	23 01	STO 1
08	51	+
09	23 02	STO 2
10	22	R↓
11	15 71	g x=0
12	13 30	GTO 30
13	01	1
14	24 01	RCL 1
15	51	+
16	23 01	STO 1
17	21	x \leftrightarrow y
18	14 51	f x>y
19	13 22	GTO 22
20	24 02	RCL 2
21	13 00	GTO 00
22	22	x \leftrightarrow y
23	24 00	RCL 0
24	51	+

DISPLAY		KEY ENTRY
LINE	CODE	
25	24 01	RCL 1
26	71	÷
27	23 61 02	STO x 2
28	22	R↓
29	13 13	GTO 13
30	01	1
31	13 00	GTO 00
32		
33		
34		
35		
36		
37		
38		
39		
40		
41		
42		
43		
44		
45		
46		
47		
48		
49		

REGISTERS
R ₀ max (n, m - n)
R ₁ Used
R ₂ Used
R ₃
R ₄
R ₅
R ₆
R ₇

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Key in m and n	m	↑				
		n	f	PRGM	R/S		mC _n
3	For new case, go to step 2.						

Examples:

- ${}_{73}C_4 = 1088430.00$
- ${}_{43}C_3 = 12341.00$

CHI-SQUARE EVALUATION

This program calculates the value of the χ^2 statistic for the goodness of fit test by the equation

$$\chi^2 = \sum_{i=1}^n \frac{(O_i - E_i)^2}{E_i}$$

where O_i = observed frequency

E_i = expected frequency.

The χ^2 statistic measures the closeness of the agreement between the observed frequencies and expected frequencies.

Notes:

1. In order to apply this test to a set of given data, it may be necessary to combine some classes to make sure that each expected frequency is not too small (say, not less than 5).
2. If the expected frequencies E_i are all equal to some value E , then E should be computed beforehand as

$$E = \frac{\sum O_i}{n}$$

and then input at each step as the expected frequency E_i .

DISPLAY		KEY ENTRY
LINE	CODE	
00		
01	00	0
02	23 00	STO 0
03	23 01	STO 1
04	74	R/S
05	23 02	STO 2
06	41	-
07	15 02	$g x^2$
08	24 02	RCL 2
09	71	\div
10	23 51 01	STO + 1
11	24 00	RCL 0
12	01	1
13	51	+
14	23 00	STO 0
15	13 04	GTO 04
16	23 02	STO 2
17	41	-
18	15 02	$g x^2$
19	24 02	RCL 2
20	71	\div
21	23 41 01	STO - 1
22	24 00	RCL 0
23	01	1
24	41	-

DISPLAY		KEY ENTRY
LINE	CODE	
25	23 00	STO 0
26	13 04	GTO 04
27		
28		
29		
30		
31		
32		
33		
34		
35		
36		
37		
38		
39		
40		
41		
42		
43		
44		
45		
46		
47		
48		
49		

REGISTERS
$R_0 n$
$R_1 \chi^2$
$R_2 E_i$
R_3
R_4
R_5
R_6
R_7

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Initialize		f	PRGM	R/S		0.00
3	Perform for $i = 1, \dots, n$:						
	Input observed and expected frequencies	O_i	\uparrow				
		E_i	R/S				i
4	Delete erroneous data	O_k	\uparrow				
		E_k	GTO	16	R/S		
5	Display χ^2		RCL	1			χ^2
6	For new case, go to step 2.						

Example:

O_i	8	50	47	56	5	14
E_i	9.6	46.75	51.85	54.4	8.25	9.15

Solution:

$$\chi^2 = 4.84$$

PAIRED t STATISTIC

Given a set of paired observations from two normal populations with means μ_1, μ_2 (unknown)

x_i	x_1	x_2	...	x_n
y_i	y_1	y_2	...	y_n

let

$$D_i = x_i - y_i$$

$$\bar{D} = \frac{1}{n} \sum_{i=1}^n D_i$$

$$s_D = \sqrt{\frac{\sum D_i^2 - \frac{1}{n} (\sum D_i)^2}{n - 1}}$$

$$s_{\bar{D}} = \frac{s_D}{\sqrt{n}}$$

The test statistic

$$t = \frac{\bar{D}}{s_{\bar{D}}},$$

which has $n - 1$ degrees of freedom (df), can be used to test the null hypothesis

$$H_0: \mu_1 = \mu_2.$$

DISPLAY		KEY ENTRY
LINE	CODE	
00		
01	41	-
02	25	$\Sigma+$
03	13 00	GTO 00
04	14 22	f s
05	24 03	RCL 3
06	14 02	f \sqrt{x}
07	71	\div
08	14 21	f \bar{x}
09	21	$x \div y$
10	71	\div
11	74	R/S
12	24 03	RCL 3
13	01	1
14	41	-
15	13 00	GTO 00
16		
17		
18		
19		
20		
21		
22		
23		
24		

DISPLAY		KEY ENTRY
LINE	CODE	
25		
26		
27		
28		
29		
30		
31		
32		
33		
34		
35		
36		
37		
38		
39		
40		
41		
42		
43		
44		
45		
46		
47		
48		
49		

REGISTERS
R ₀
R ₁
R ₂
R ₃ n
R ₄ Used
R ₅ Used
R ₆ ΣD_i
R ₇ ΣD_i^2

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Initialize		f	REG	f	PRGM	
3	Perform for $i = 1, \dots, n$:						
	Input one pair of observations	x_i	\uparrow				
		y_i	R/S				i
4	Delete erroneous data	x_k	\uparrow				
		y_k	-	f	$\Sigma-$		
5	Compute t and df		GTO	04	R/S		t
			R/S				df
6	For new case, go to step 2.						

Example:

x_i	14	17.5	17	17.5	15.4
y_i	17	20.7	21.6	20.9	17.2

Solution:

$$t = -7.16$$

$$df = 4.00$$

t STATISTIC FOR TWO MEANS

Suppose $\{x_1, x_2, \dots, x_{n_1}\}$ and $\{y_1, y_2, \dots, y_{n_2}\}$ are independent random samples from two normal populations having means μ_1, μ_2 (unknown) and the same unknown variance σ^2 .

We want to test the null hypothesis

$$H_0: \mu_1 - \mu_2 = D$$

where D is a given number.

Define

$$\bar{x} = \frac{1}{n_1} \sum_{i=1}^{n_1} x_i$$

$$\bar{y} = \frac{1}{n_2} \sum_{i=1}^{n_2} y_i$$

$$t = \frac{\bar{x} - \bar{y} - D}{\sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \sqrt{\frac{\sum x_i^2 - n_1 \bar{x}^2 + \sum y_i^2 - n_2 \bar{y}^2}{n_1 + n_2 - 2}}}$$

We can use this t statistic, which has the t distribution with $n_1 + n_2 - 2$ degrees of freedom, to test the null hypothesis H_0 .

DISPLAY		KEY ENTRY
LINE	CODE	
00		
01	24 03	RCL 3
02	23 00	STO 0
03	24 06	RCL 6
04	23 01	STO 1
05	14 21	$f\bar{x}$
06	23 02	STO 2
07	34	CLX
08	23 03	STO 3
09	23 06	STO 6
10	23 07	STO 7
11	74	R/S
12	31	\uparrow
13	14 21	$f\bar{x}$
14	51	+
15	24 02	RCL 2
16	21	$x\bar{z}y$
17	41	-
18	24 00	RCL 0
19	15 22	$g\ 1/x$
20	24 03	RCL 3
21	15 22	$g\ 1/x$
22	51	+
23	14 02	$f\sqrt{x}$
24	71	\div

DISPLAY		KEY ENTRY
LINE	CODE	
25	24 01	RCL 1
26	24 02	RCL 2
27	15 02	$g\ x^2$
28	24 00	RCL 0
29	61	x
30	41	-
31	24 06	RCL 6
32	51	+
33	14 21	$f\bar{x}$
34	15 02	$g\ x^2$
35	24 03	RCL 3
36	61	x
37	41	-
38	24 00	RCL 0
39	24 03	RCL 3
40	51	+
41	02	2
42	41	-
43	71	\div
44	14 02	$f\sqrt{x}$
45	71	\div
46	13 00	GTO 00
47		
48		
49		

REGISTERS
$R_0\ n_1$
$R_1\ \Sigma x^2$
$R_2\ \bar{x}$
$R_3\ n_2$
R_4 Used
R_5 Used
$R_6\ \Sigma y^2$
$R_7\ \Sigma y$

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Initialize		f	REG			
3	Perform for $i = 1, \dots, n_1$:						
	Input x-value	x_i	$\Sigma+$				i
4	Initialize for y		f	PRGM	R/S		0.00
5	Perform for $i = 1, \dots, n_2$:						
	Input y-value	y_i	$\Sigma+$				i
6	Input D and compute t	D	R/S				t
7	To find the means of x- and y-values						
			RCL	2			\bar{x}
			f	\bar{x}			\bar{y}
8	For a new case, go to step 2.						

Example:

x: 79, 84, 108, 114, 120, 103, 122, 120

y: 91, 103, 90, 113, 108, 87, 100, 80, 99, 54

$$n_1 = 8$$

$$n_2 = 10$$

$$D = 0 \text{ (i.e., } H_0: \mu_1 = \mu_2 \text{)}$$

Solution:

$$t = 1.73$$

$$\bar{x} = 106.25$$

$$\bar{y} = 92.50$$

ONE SAMPLE TEST STATISTICS FOR THE MEAN

For a normal population (x_1, x_2, \dots, x_n) with a known variance σ^2 , a test of the null hypothesis

$$H_0: \text{mean } \mu = \mu_0$$

is based on the z statistic (which has a standard normal distribution)

$$z = \frac{\sqrt{n} (\bar{x} - \mu_0)}{\sigma}$$

If the variance σ^2 is unknown, then

$$t = \frac{\sqrt{n} (\bar{x} - \mu_0)}{s}$$

is used instead. This t statistic has the t distribution with $n - 1$ degrees of freedom. \bar{x} and s are the sample mean and standard deviation.

DISPLAY		KEY ENTRY
LINE	CODE	
00		
01	14 21	f \bar{x}
02	21	$x \div y$
03	41	-
04	24 03	RCL 3
05	14 02	f \sqrt{x}
06	61	x
07	23 00	STO 0
08	34	CLX
09	74	R/S
10	24 00	RCL 0
11	14 22	f s
12	71	\div
13	74	R/S
14	24 00	RCL 0
15	21	$x \div y$
16	71	\div
17	13 00	GTO 00
18		
19		
20		
21		
22		
23		
24		

DISPLAY		KEY ENTRY
LINE	CODE	
25		
26		
27		
28		
29		
30		
31		
32		
33		
34		
35		
36		
37		
38		
39		
40		
41		
42		
43		
44		
45		
46		
47		
48		
49		

REGISTERS	
R ₀	$\sqrt{n} (\bar{x} - \mu_0)$
R ₁	
R ₂	
R ₃	n
R ₄	Used
R ₅	Used
R ₆	Σx
R ₇	Σx^2

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Initialize		f	REG			
3	Perform for $i = 1, \dots, n$						
	Input value	x_i	$\Sigma+$				i
4	Input μ_0	μ_0	f	PRGM	R/S		0.00
5	Compute t		GTO	10	R/S		t
	or						
5	Input σ and compute z	σ	GTO	14	R/S		z
6	For new case, go to step 2.						

Example:

Suppose $\mu_0 = 2$, for the following set of data

$\{2.73, 0.45, 2.52, 1.19, 3.51, 2.75, 1.79, 1.83, 1, 0.87, 1.9, 1.62, 1.74, 1.92, 1.24, 2.68\}$

Solution:

test statistic $t = -.69$

or $z = -.57$ if $\sigma = 1$.

CHAPTER 7 SURVEYING

FIELD ANGLE TRAVERSE

A traverse is a series of line segments joined with specific lengths and angular relations to each other. With many applications in surveying, the field angle traverse may be used in establishing boundary lines, road layout, and in numerous construction situations. The transit and "chain" (commonly a length of steel tape) are often used to establish the angles and distances involved in a field angle traverse.

Starting at a known point from a given reference direction, the transit man establishes the direction of a new line by measuring the angle or deflection turned to align the scope of the transit to the new line. With a measured distance to the end point on the new line and its direction, coordinates of the end point relative to the origin may be established. The transit is then moved to the new "origin", the reference direction is the line just determined, and the process continues.

To run this program, the user must input the northing and easting of his starting point, the reference azimuth, and then the direction and distance from each point in the traverse to the next point. The direction may be input either as a deflection right or left, or as an angle right or left. The distance may be input either as horizontal distance, or as slope distance with zenith angle.

Equations:

$$H \text{ Dist} = S \text{ Dist} \sin (Zn\theta \text{ ang})$$

$$N_{i+1} = N_i + H \text{ Dist} \cos Az$$

$$E_{i+1} = E_i + H \text{ Dist} \sin Az$$

$$\begin{aligned} \text{Area} = & \frac{1}{2} [(N_2 + N_1) (E_2 - E_1) + (N_3 + N_2) (E_3 - E_2) + \\ & \dots + (N_n + N_1) (E_1 - E_n)] \end{aligned}$$

where: N, E = Northing, easting of a point

Subscript i refers to current point

Subscript n refers to next to last point

Numeric subscript refers to point number

Az = Azimuth of a course

H Dist = Horizontal distance

S Dist = Slope distance

Znth ang = Zenith angle

Notes:

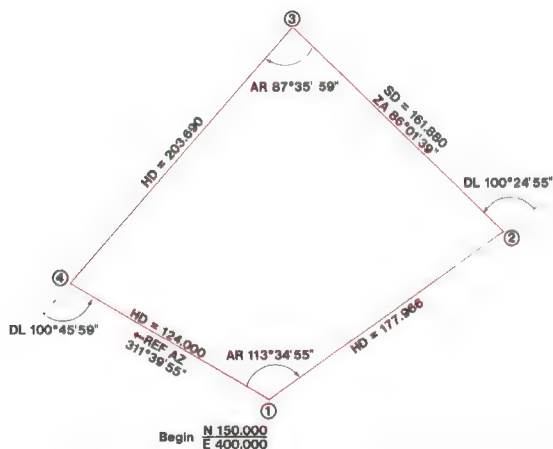
1. The calculation for area of a closed traverse may be inaccurate for cases in which the coordinates of the figure are quite large, such as in state plane coordinate systems. In such cases, the user may run the Area By Double Meridian Distance program to calculate an accurate value for area once the bearings and distances have been established by this program.
2. All angular inputs and outputs are in the form degrees, minutes, and seconds (D.MS).

DISPLAY		KEY ENTRY	X	Y	Z	T	COMMENTS	REGISTERS
LINE	CODE							
00								
01	15 00	g → H	Ref Az				Convert to decimal degrees	R 0 Az
02	01	1	1	Ref Az				
03	08	8	18	Ref Az				R 1 Current
04	00	0	180	Ref Az				N
05	51	+	180 + Az					
06	23 00	STO 0	180 + Az					R 2 Current
07	24 01	RCL 1	N _i	180 + Az				E
08	23 05	STO 5	N _i	180 + Az			Initialize "previous N"	
09	00	0	0	N _i	180 + Az		Clear R ₃ , R ₄ , for	R 3 Σ H Dist
10	23 03	STO 3	0	N _i	180 + Az		accumulation	
11	23 04	STO 4	0	N _i	180 + Az			
12	74	R/S	0	N _i	180 + Az			R 4 Area
13	15 00	g → H	Angle				Convert to decimal degrees	
14	01	1	1	Angle				
15	08	8	18	Angle				R 5 Previous
16	00	0	180	Angle				N
17	51	+	180 + Ang					
18	14 00	f → H.MS	(D.MS)					
19	15 00	g → H	Defl				Deflection comes in here	R 6
20	24 00	RCL 0	Az	Defl				
21	51	+	Az + Defl				Find new azimuth	R 7
22	23 00	STO 0	Az _i					
23	14 00	f → H.MS	Az _i				Convert to D.MS for	
24	74	R/S	Az _i				display	
25	13 29	GTO 29	H Dist					
26	21	x ² y	Zn Ang	S Dist				
27	14 04	f SIN	sin Zn	S Dist				
28	61	x	H Dist				H Dist = sin Zn (S Dist)	
29	23 51 03	STO + 3	H Dist				Accumulate H Dist	
30	24 00	RCL 0	Az	H Dist				
31	21	x ² y	H Dist	Az				
32	14 09	f → R	ΔN	ΔE				
33	23 51 01	STO + 1	ΔN	ΔE			ΔN = H Dist (cos Az)	
34	21	x ² y	ΔE	ΔN				
35	23 51 02	STO + 2	ΔE	ΔN			ΔE = H Dist (sin Az)	
36	24 05	RCL 5	N _{i-1}	ΔE	ΔN			
37	24 01	RCL 1	N _i	N _{i-1}	ΔE	ΔN		
38	23 05	STO 5	N _i	N _{i-1}	ΔE	ΔN	Update "previous N"	
39	51	+	(N _i + N _{i-1})	ΔE	ΔN			
40	61	x	ΔA	ΔN			ΔA = (N _i + N _{i-1}) ΔE	
41	02	2	ΔA	ΔN				
42	71	÷	1/2 ΔA	ΔN				
43	23 51 04	STO + 4	1/2 ΔA	ΔN			Accumulate Area	
44	24 01	RCL 1	N _i	1/2 ΔA	ΔN			
45	74	R/S	N _i	1/2 ΔA	ΔN		Display Northing	
46	24 02	RCL 2	E _i	N _i	1/2 ΔA	ΔN		
47	13 12	GTO 12	E _i	N _i	1/2 ΔA	ΔN	Display Easting	
48								
49								

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Input the starting point coordinates						
		N_i	STO	1			
		E_i	STO	2			
3	Input the reference azimuth	Ref Az, D.MS	f	PRGM	R/S		0.00
4a.	If angle right	AR, D.MS	R/S				Az_i , D.MS
4b.	If angle left	AL, D.MS	CHS	R/S			Az_i , D.MS
4c.	If deflection right	DR, D.MS	GTO	19	R/S		Az_i , D.MS
4d.	If deflection left	DL, D.MS	CHS	GTO	19	R/S	Az_i , D.MS
5a.	If horizontal distance	H Dist	R/S				N_i
			R/S				E_i
5b.	If slope distance, input Zenith						
	Angle and Slope Distance	Z_n , Ang, D.MS	↑				
		S Dist	GTO	26	R/S		N_i
			R/S				E_i
6	Repeat steps 4-5 for successive courses.						
7	Display total horizontal distance traversed		RCL	3			Σ H Dist
8	Display area for closed traverse (ignore sign)		RCL	4			Area

Example:

The diagram below shows measurements taken for a closed traverse. Find the coordinates of points 2, 3, and 4, the total horizontal distance traversed, and the area of the figure.



Solution:

150 **STO** **1** 400 **STO** **2** 311.3955 **f** **PRGM** **R/S** → 0.00

113.3455 **R/S** 177.966 **R/S** → 224.515 (N_2)

R/S → 561.615 (E_2)

100.2455 **CHS** **GTO** **1** **9** **R/S** 86.0139 **↑** 161.880 **GTO**

2 **6** **R/S** → 356.526 (N_3)

R/S → 468.601 (E_3)

87.3559 **R/S** 203.690 **R/S** → 232.335 (N_4)

R/S → 307.151 (E_4)

100.4559 **CHS** **GTO** **1** **9** **R/S** 124.0 **R/S** → 149.903 (N_1)

R/S → 399.784 (E_1)

RCL **3** → 667.144 (ΣH
Dist)

RCL **4** → 26590.68
(Area)

AREA BY DOUBLE MERIDIAN DISTANCE

This program computes the area of a straight-sided closed figure from the bearings and lengths of its sides. It is generally more accurate than methods which calculate area from the coordinates of the figure.

$$\text{Area} = \frac{1}{2} \sum_i \text{DMD}_i \times \text{Latitude}_i$$

$$\text{DMD}_i = \text{DMD}_{i-1} + \text{Departure}_{i-1} + \text{Departure}_i$$

where

$$\text{Departure}_i = \text{Dist}_i \sin \text{Az}_i$$

$$\text{Latitude}_i = \text{Dist}_i \cos \text{Az}_i$$

Note:

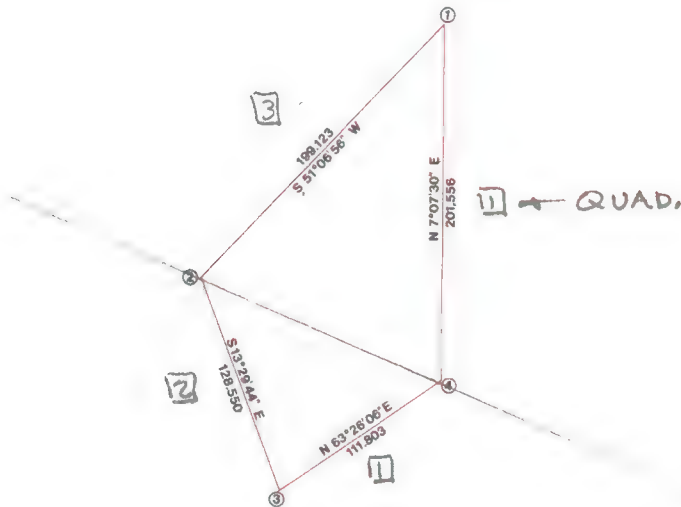
Angles are input as bearing and quadrant code. The quadrant code is 1 for NE, 2 for SE, 3 for SW, and 4 for NW.

DISPLAY			KEY ENTRY	DISPLAY			KEY ENTRY	REGISTERS	
LINE	CODE			LINE	CODE				
00				25	21	$x \leftrightarrow y$		R ₀	Az _i
01	15 00	g → H		26	14 09	f → R		R ₁	DMD _{i-1}
02	74	R/S		27	21	$x \leftrightarrow y$		R ₂	Departure _{i-1}
03	02	2		28	24 02	RCL 2		R ₃	Area
04	71	÷		29	21	$x \leftrightarrow y$		R ₄	
05	31	↑		30	23 02	STO 2		R ₅	
06	14 01	f INT		31	51	+		R ₆	
07	14 61	f x ≠ y		32	24 01	RCL 1		R ₇	
08	13 15	GTO 15		33	51	+			
09	22	R↓		34	23 01	STO 1			
10	22	R↓		35	61	x			
11	32	CHS		36	02	2			
12	22	R↓		37	71	÷			
13	22	R↓		38	23 51 03	STO + 3			
14	22	R↓		39	24 03	RCL 3			
15	14 01	f INT		40	13 00	GTO 00			
16	01	1		41					
17	08	8		42					
18	00	0		43					
19	61	x		44					
20	51	+		45					
21	23 00	STO 0		46					
22	14 00	f → H.MS		47					
23	74	R/S		48					
24	24 00	RCL 0		49					

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Initialize		f	REG	f	PRGM	
3	Input bearing	Brg, D.MS	R/S				Brg, dec. deg.
4	Input quadrant code	Quad	R/S				Az, D.MS
5	Input distance	Dist	R/S				Area
6	Repeat steps 3, 4, 5 for successive courses. Area is displayed after last distance has been input.						

Example:

Compute the area of the figure below.

**Solution:**

Area = 20937.44 sq. ft.

✓ PROGRAM ERROR.

CORRECTIONS:

LINE 02.5 INSERT ENTER ↑

LINE ~~08~~ 09 CHANGE TO GTO 15

INVERSE FROM COORDINATES

This program uses coordinates to calculate distance and bearing between points of a traverse. The area in square feet and a summation of distance inversed are also computed.

$$H \text{ Dist} = \sqrt{(N_i - N_{i-1})^2 + (E_i - E_{i-1})^2}$$

$$Az = \tan^{-1} \frac{E_i - E_{i-1}}{N_i - N_{i-1}}$$

$$\begin{aligned} \text{Area} = & \frac{1}{2}[(N_2 + N_1)(E_2 - E_1) + (N_3 + N_2)(E_3 - E_2) + \\ & \dots (N_n + N_1)(E_1 - E_n)] \end{aligned}$$

where N, E = Northing, easting of a point

Subscript i refers to current point

Subscript n refers to next to last point

Numeric subscript refers to point number

H Dist = Horizontal distance

Az = Azimuth of a course

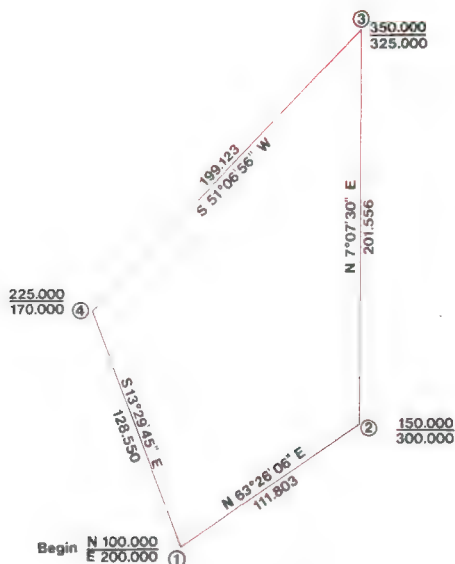
DISPLAY		KEY ENTRY
LINE	CODE	
00		
01	14 33	f REG
02	23 02	STO 2
03	21	x \leftrightarrow y
04	23 00	STO 0
05	23 01	STO 1
06	74	R/S
07	24 02	RCL 2
08	41	-
09	23 51 02	STO + 2
10	23 05	STO 5
11	21	x \leftrightarrow y
12	24 01	RCL 1
13	41	-
14	23 51 01	STO + 1
15	15 09	g \rightarrow P
16	23 51 03	STO + 3
17	74	R/S
18	21	x \leftrightarrow y
19	15 51	g x \geq 0
20	13 25	GTO 25
21	03	3
22	06	6
23	00	0
24	51	+

DISPLAY		KEY ENTRY
LINE	CODE	
25	31	\uparrow
26	31	\uparrow
27	09	9
28	00	0
29	71	\div
30	01	1
31	51	+
32	14 01	f INT
33	21	x \leftrightarrow y
34	14 04	f SIN
35	15 04	g SIN $^{-1}$
36	15 41	g x<0
37	32	CHS
38	14 00	f \rightarrow H.MS
39	24 00	RCL 0
40	24 01	RCL 1
41	23 00	STO 0
42	51	+
43	24 05	RCL 5
44	61	x
45	02	2
46	71	\div
47	23 51 04	STO + 4
48	22	\uparrow
49	13 06	GTO 06

REGISTERS	
R ₀	Previous N
R ₁	Current N
R ₂	Current E
R ₃	Σ H Dist
R ₄	Area
R ₅	ΔE
R ₆	
R ₇	

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Input starting coordinates	N_1	\uparrow				
		E_1	f	PRGM	R/S		
3	Input next coordinates and						
	display distance	N_i	\uparrow				
		E_i	R/S				H Dist
4	Compute bearing and quadrant						
	code		R/S				Brg, D.MS
			R↓				Quad code
5	Repeat steps 3-4 for successive						
	courses						
6	Display total distance inversed		RCL	3			Σ H Dist
7	Display area of closed figure						
	(ignore the sign)		RCL	4			Area

Example:



Area = 20937.5 Sq. ft.

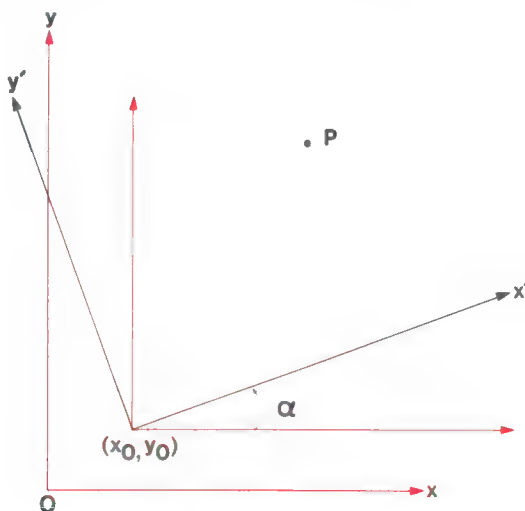
Total distance inversed = 641.033

CHAPTER 8

TRIGONOMETRY AND ANALYTICAL GEOMETRY

COORDINATE TRANSLATION AND ROTATION

There are occasions, for instance in cartography or metalworking, when it is necessary or advantageous to shift one's frame of reference. In mathematical terms, the occasion calls for a translation and/or rotation of the coordinate system. The origin is translated from $(0, 0)$ to a new point, (x_0, y_0) , and the x and y axes are then rotated through an angle α to give new axes, x' and y' . Suppose that a point P has coordinates (x, y) with respect to the old system of x and y axes. The problem then is to find the coordinates (x', y') of P with respect to the new system whose axes are x' and y' . The diagram below illustrates this situation.



Equations:

$$x' = (x - x_0) \cos \alpha + (y - y_0) \sin \alpha$$

$$y' = -(x - x_0) \sin \alpha + (y - y_0) \cos \alpha$$

Notes:

1. The program may be used to solve a problem of translation only, or of rotation only, or of combined translation and rotation. If the problem involves translation alone, a value of $\alpha = 0$ must be input. For rotation alone, the values $x_0 = y_0 = 0$ must be input.
2. The program assumes the following sign convention: α should be input as a positive number if the rotation is counterclockwise, and negative if clockwise.

Programming Remarks:

This program demonstrates a particularly powerful application of the polar-to-rectangular conversion ($\boxed{f} \rightarrow \boxed{R}$) when combined with the capabilities of the four-register stack. The subterms $(x - x_0) \cos \alpha$, $(x - x_0) \sin \alpha$, $(y - y_0) \cos \alpha$, and $(y - y_0) \sin \alpha$ are all generated through $\boxed{f} \rightarrow \boxed{R}$ and stored in the stack until needed. A more straightforward program using $\boxed{f} \boxed{\sin}$ and $\boxed{f} \boxed{\cos}$ would have required 30 program steps (as compared to 19) and one more storage register.

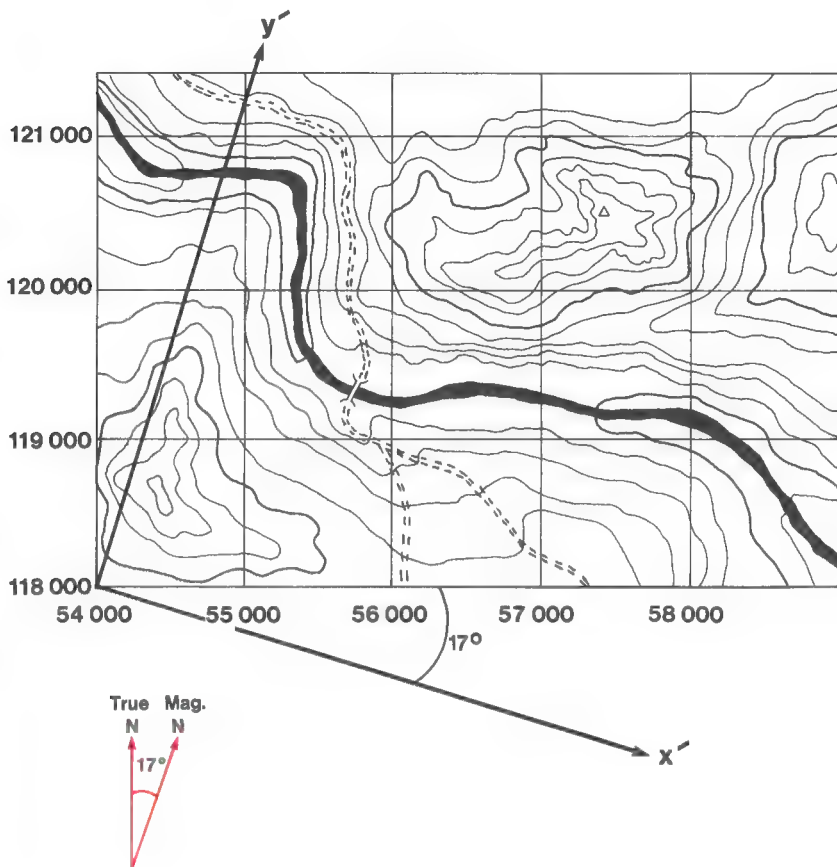
DISPLAY		KEY ENTRY	X	Y	Z	T	COMMENTS	REGISTERS
LINE	CODE							
00			y	x				R 0 x_0
01	23 03	STO 3	y	x				
02	22	R↓	x			y		
03	24 02	RCL 2	α	x				R 1 y_0
04	21	$x \leftrightarrow y$	x	α				
05	24 00	RCL 0	x_0	x	α			
06	41	-	Δx	α			$\Delta x = x - x_0$	R 2 α
07	14 09	f → R	$\Delta x \cos \alpha$	$\Delta x \sin \alpha$				
08	24 03	RCL 3	y	$\Delta x \cos \alpha$	$\Delta x \sin \alpha$			
09	24 01	RCL 1	y_0	y	$\Delta x \cos \alpha$	$\Delta x \sin \alpha$		R 3 y
10	41	-	Δy	$\Delta x \cos \alpha$	$\Delta x \sin \alpha$	$\Delta x \sin \alpha$	$\Delta y = y - y_0$	
11	24 02	RCL 2	α	Δy	$\Delta x \cos \alpha$	$\Delta x \sin \alpha$		
12	21	$x \leftrightarrow y$	Δy	α	$\Delta x \cos \alpha$	$\Delta x \sin \alpha$		R 4
13	14 09	f → R	$\Delta y \cos \alpha$	$\Delta y \sin \alpha$	$\Delta x \cos \alpha$	$\Delta x \sin \alpha$		
14	22	R↓	$\Delta y \sin \alpha$	$\Delta x \cos \alpha$	$\Delta x \sin \alpha$	$\Delta y \cos \alpha$		
15	51	+	x'	$\Delta x \sin \alpha$	$\Delta y \cos \alpha$	$\Delta y \sin \alpha$	$x' = \Delta x \cos \alpha + \Delta y \sin \alpha$	R 5
16	74	R/S	x'	$\Delta x \sin \alpha$	$\Delta y \cos \alpha$	$\Delta y \sin \alpha$		
17	22	R↓	$\Delta x \sin \alpha$	$\Delta y \cos \alpha$	$\Delta y \sin \alpha$	x'		
18	41	-	y'	$\Delta y \cos \alpha$	x'	x'	$y' = -\Delta x \sin \alpha + \Delta y \cos \alpha$	R 6
19	13 00	GTO 00	y'	$\Delta y \cos \alpha$	x'	x'		R 7
20								
21								
22								
23								
24								
25								
26								
27								
28								
29								
30								
31								
32								
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48								
49								

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Store origin of new coordinate system	x_0	STO	0			
		y_0	STO	1			
3	Store angle of rotation	α	STO	2	f	PRGM	
4	Convert coordinates from old to new system	x	\uparrow				
		y	R/S				x'
			R/S				y'
5	Perform step 4 for as many points as desired						
6	For a new case, go to step 2.						

Example:

A backpacker's route will take him cross-country away from the marked trails of an area. He knows that he will have to check his compass frequently against his map over this terrain, and regrets that the map is in such an inconvenient format for his purposes. In the first place, the grid lines on his map represent distances in feet from an origin about 25 miles away, which are such large numbers that they are hard to calculate with. Secondly, the map's grid is based on true north while his compass readings are relative to magnetic north, a variation of 17° .

Before he leaves home, the packer decides to draw a rough version of the map for his own convenience, locating his origin at the grid point (54 000, 118 000) and rotating his axes by 17° in a clockwise direction. As a first step, he wants to find the new coordinates of the bridge and the peak of the hill, whose coordinates in the old system are (55 750, 119 300) and (57 450, 120 500) respectively.



Solution:

54000 **STO** **0** 118000 **STO** **1** 17 **CHS** **STO** **2** **f** **PRGM**

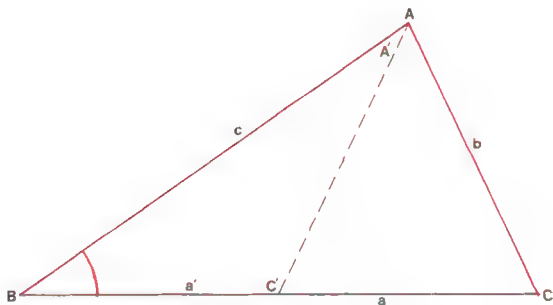
55750 **+** 119300 **R/S** \longrightarrow 1293.45
R/S \longrightarrow 1754.85

The new coordinates of the bridge are (1293, 1755).

57450 **+** 120500 **R/S** \longrightarrow 2568.32
R/S \longrightarrow 3399.44

The new coordinates of the peak are (2568, 3399).

TRIANGLE SOLUTION B, b, c



Given two sides and a non-included angle, this program solves the triangle for the remaining parameters by the following formulas:

1. $C = \sin^{-1} \left(\frac{c \sin B}{b} \right)$
2. $A = 2 \sin^{-1} 1 - (B + C) = \pi \text{ radians} - (B + C) = 180^\circ - (B + C)$
 $= 200 \text{ grads} - (B + C)$
3. $a = \frac{b \sin A}{\sin B}$

If B is acute ($< 90^\circ$) and $b < c$, a second set of solutions exists and is calculated by the following formulas:

4. $C' = 2 \sin^{-1} 1 - C$
5. $A' = 2 \sin^{-1} 1 - (B + C')$
6. $a' = \frac{b \sin A'}{\sin B}$

The area is computed with the formula

$$\text{Area} = \frac{1}{2} ac \sin B$$

This program works in any angular mode. However, if in degrees, decimal degrees are assumed.

DISPLAY		KEY ENTRY
LINE	CODE	
00		
01	24 03	RCL 3
02	24 01	RCL 1
03	14 04	f SIN
04	61	x
05	24 02	RCL 2
06	71	÷
07	15 04	g SIN ⁻¹
08	23 05	STO 5
09	74	R/S
10	24 01	RCL 1
11	51	+
12	01	1
13	15 04	g SIN ⁻¹
14	02	2
15	61	x
16	23 04	STO 4
17	21	x ² y
18	41	-
19	74	R/S
20	14 04	f SIN
21	24 02	RCL 2
22	61	x
23	24 01	RCL 1
24	14 04	f SIN

DISPLAY		KEY ENTRY
LINE	CODE	
25	71	÷
26	74	R/S
27	24 03	RCL 3
28	61	x
29	24 01	RCL 1
30	14 04	f SIN
31	61	x
32	02	2
33	71	÷
34	74	R/S
35	24 04	RCL 4
36	24 05	RCL 5
37	41	-
38	74	R/S
39	13 10	GTO 10
40		
41		
42		
43		
44		
45		
46		
47		
48		
49		

REGISTERS
R ₀
R ₁ B
R ₂ b
R ₃ c
R ₄ 2 sin ⁻¹ 1
R ₅ C
R ₆
R ₇

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Store B, b, and c	B	STO	1			
		b	STO	2			
		c	STO	3			
3	Solve the triangle		f	PRGM	R/S		C*
			R/S				A*
			R/S				a*
			R/S				Area
4	If $B < 90^\circ$ and $b < c$, find al-						
	ternate solution		R/S				C'*
			R/S				A'*
			R/S				a'*
			R/S				Area'
	* The stack must be maintained						
	at these positions.						

Example:

Given the following two sides and non-included angle:

$$B = 42.3^\circ$$

$$b = 25.6$$

$$c = 32.8$$

Solve the triangle.

Solution:

Since B is less than 90° and $b < c$, two sets of solutions exist.

$$C = 59.58^\circ$$

$$A = 78.12^\circ$$

$$a = 37.22$$

$$\text{Area} = 410.85$$

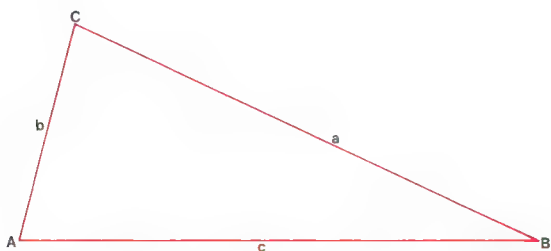
$$C' = 120.42^\circ$$

$$A' = 17.28^\circ$$

$$a' = 11.30$$

$$\text{Area}' = 124.68$$

TRIANGLE SOLUTION a, b, c



Given three sides of a triangle this program solves the triangle for the remaining parameters by the following formulas:

$$C = \cos^{-1} \left(\frac{a^2 + b^2 - c^2}{2ab} \right)$$

$$B = \sin^{-1} \left(\frac{b \sin C}{c} \right) \quad A = \sin^{-1} \left(\frac{a \sin C}{c} \right)$$

This program also computes the area by the following formula:

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

$$\text{where } s = \frac{1}{2} (a + b + c)$$

Reletter if necessary to make c the largest side. The program works in any angular mode. However, if in degree mode decimal degrees are assumed.

DISPLAY		KEY ENTRY
LINE	CODE	
00		
01	24 01	RCL 1
02	24 02	RCL 2
03	15 09	$g \rightarrow P$
04	15 02	$g x^2$
05	24 03	RCL 3
06	15 02	$g x^2$
07	41	-
08	24 01	RCL 1
09	24 02	RCL 2
10	61	x
11	02	2
12	61	x
13	71	\div
14	15 05	$g \cos^{-1}$
15	74	R/S
16	14 04	f SIN
17	24 03	RCL 3
18	71	\div
19	23 00	STO 0
20	24 02	RCL 2
21	61	x
22	15 04	$g \sin^{-1}$
23	74	R/S
24	24 00	RCL 0

DISPLAY		KEY ENTRY
LINE	CODE	
25	24 01	RCL 1
26	61	x
27	15 05	$g \sin^{-1}$
28	74	R/S
29	24 01	RCL 1
30	24 02	RCL 2
31	51	+
32	24 03	RCL 3
33	51	+
34	02	2
35	71	\div
36	31	\uparrow
37	23 00	STO 0
38	24 01	RCL 1
39	41	-
40	61	x
41	24 00	RCL 0
42	24 02	RCL 2
43	41	-
44	61	x
45	24 00	RCL 0
46	24 03	RCL 3
47	41	-
48	61	x
49	14 02	$f\sqrt{x}$

REGISTERS
R ₀ Used
R ₁ a
R ₂ b
R ₃ c
R ₄
R ₅
R ₆
R ₇

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Store sides (c is the largest)	a	STO	1			
		b	STO	2			
		c	STO	3			
3	Solve the triangle		f	PRGM	R/S		C°
			R/S				B°
			R/S				A
			R/S				Area
4	If only the area is needed:	a	STO	1			
		b	STO	2			
		c	STO	3			
			GTO	29	R/S		Area
	* The stack must be maintained						
	at these points.						

Example:

Let $a = 5.43$, $b = 10.46$, $c = 14.87$

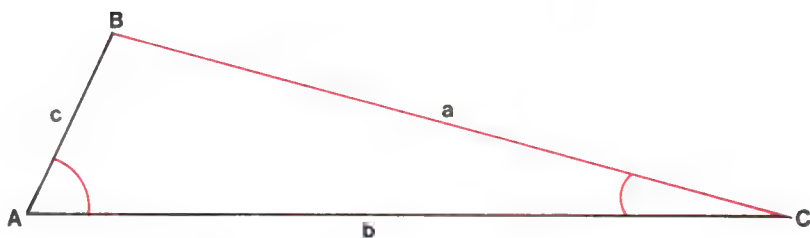
Solution:

$$C = 136.37^\circ$$

$$B = 29.04^\circ$$

$$A = 14.59^\circ$$

$$\text{Area} = 19.60$$

TRIANGLE SOLUTION a, A, C 

Given two angles and an opposite side this program solves the triangle for the remaining parameters by the following formulas:

$$\begin{aligned} B &= 2 \sin^{-1} 1 - (A + C) = \pi \text{ radians} - (A + C) = 180^\circ - (A + C) \\ &= 200 \text{ grads} - (A + C) \end{aligned}$$

$$b = \frac{a \sin B}{\sin A}$$

$$c = \frac{a \sin C}{\sin A}$$

The area is computed with the following formula:

$$\text{Area} = \frac{1}{2} ab \sin C$$

The program works in any angular mode. However, if in degree mode all angles are assumed to be in decimal degrees.

DISPLAY		KEY ENTRY
LINE	CODE	
00		
01	01	1
02	15 04	$g \sin^{-1}$
03	02	2
04	61	x
05	24 02	RCL 2
06	24 03	RCL 3
07	51	+
08	41	-
09	74	R/S
10	14 04	f SIN
11	24 01	RCL 1
12	61	x
13	24 02	RCL 2
14	14 04	f SIN
15	71	\div
16	23 04	STO 4
17	74	R/S
18	24 01	RCL 1
19	14 73	f LASTx
20	71	\div
21	24 03	RCL 3
22	14 04	f SIN
23	61	x
24	74	R/S

DISPLAY		KEY ENTRY
LINE	CODE	
25	24 01	RCL 1
26	24 04	RCL 4
27	61	x
28	24 03	RCL 3
29	14 04	f SIN
30	61	x
31	02	2
32	71	\div
33	13 00	GTO 00
34		
35		
36		
37		
38		
39		
40		
41		
42		
43		
44		
45		
46		
47		
48		
49		

REGISTERS	
R ₀	
R ₁ a	
R ₂ A	
R ₃ C	
R ₄ b	
R ₅	
R ₆	
R ₇	

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Store a, A, and C	a	STO	1			
		A	STO	2			
		C	STO	3			
3	Solve the triangle		f	PRGM	R/S		B*
			R/S				b*
			R/S				c
			R/S				Area
	* The stack must be maintained						
	at these points.						

Example:

Let $a = 19.6$, $A = 40.25^\circ$, $C = 61.06^\circ$

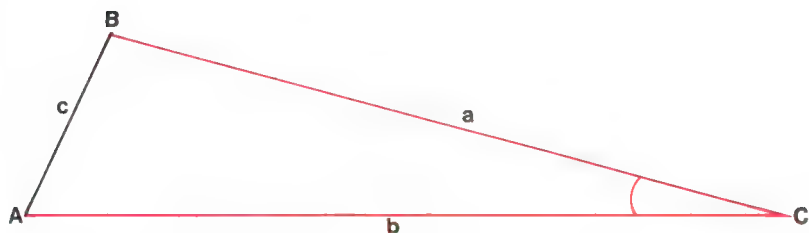
Solution:

$$B = 78.69^\circ$$

$$b = 29.75$$

$$c = 26.55$$

$$\text{Area} = 255.11$$

TRIANGLE SOLUTION a, b, C 

Given two sides and their included angle this program solves the triangle for the remaining parameters by the following formulas:

$$c = \sqrt{a^2 + b^2 - 2ab \cos C} \qquad A = \sin^{-1} \left(\frac{a \sin C}{c} \right)$$

$$\begin{aligned} B &= 2 \sin^{-1} 1 - (A + C) = \pi \text{ radians} - (A + C) = 180^\circ - (A + C) \\ &= 200 \text{ grads} - (A + C) \end{aligned}$$

The area is calculated by

$$\text{Area} = \frac{1}{2} ab \sin C$$

Reletter if necessary, to make a the smaller of a and b.

This program works in any angular mode. However, if in degrees decimal degrees are assumed.

DISPLAY		KEY ENTRY
LINE	CODE	
00		
01	24 01	RCL 1
02	24 02	RCL 2
03	15 09	$g \rightarrow P$
04	15 02	$g x^2$
05	24 01	RCL 1
06	24 02	RCL 2
07	61	x
08	02	2
09	61	x
10	24 03	RCL 3
11	14 05	f COS
12	61	x
13	41	-
14	14 02	$f\sqrt{x}$
15	74	R/S
16	24 01	RCL 1
17	24 03	RCL 3
18	14 04	f SIN
19	61	x
20	21	$x \leftrightarrow y$
21	71	\div
22	15 04	$g \sin^{-1}$
23	74	R/S
24	01	1

DISPLAY		KEY ENTRY
LINE	CODE	
25	15 04	$g \sin^{-1}$
26	02	2
27	61	x
28	21	$x \leftrightarrow y$
29	24 03	RCL 3
30	51	+
31	41	-
32	74	R/S
33	24 03	RCL 3
34	14 04	f SIN
35	24 01	RCL 1
36	61	x
37	24 02	RCL 2
38	61	x
39	02	2
40	71	\div
41	13 00	GTO 00
42		
43		
44		
45		
46		
47		
48		
49		

REGISTERS	
R ₀	
R ₁ a	
R ₂ b	
R ₃ C	
R ₄	
R ₅	
R ₆	
R ₇	

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Store a, b, and C (a is smaller of a and b)	a	STO	1			
		b	STO	2			
		C	STO	3			
3	Solve the triangle		f	PRGM	R/S		c*
			R/S				A*
			R/S				B
			R/S				Area
4	If only the area is needed:	a	STO	1			
		b	STO	2			
		C	STO	3			
			GTO	33	R/S		Area
	* The stack must be maintained at these points.						

Example:

Let $a = 146$, $b = 227$, $C = 31.49^\circ$

Solution:

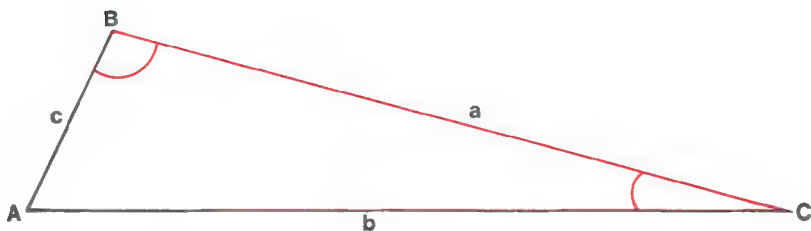
$$c = 127.76$$

$$A = 36.65^\circ$$

$$B = 111.86^\circ$$

$$\text{Area} = 8655.86$$

TRIANGLE SOLUTION a, B, C



Given two angles and their included side this program solves the triangle for the remaining parameters by the following formulas:

$$A = 2 \sin^{-1} 1 - (B + C) = \pi \text{ radians} - (B + C) = 180^\circ - (B + C)$$

$$= 200 \text{ grads} - (B + C)$$

$$b = \frac{a \sin B}{\sin A}$$

$$c = \frac{a \sin C}{\sin A}$$

The area is found using the formula:

$$\text{Area} = \frac{a^2 \sin B \sin C}{2 \sin (B + C)}$$

The program works in any angular mode. However, if in degrees the program assumes decimal degrees.

DISPLAY		KEY ENTRY
LINE	CODE	
00		
01	01	1
02	15 04	$g \sin^{-1}$
03	02	2
04	61	x
05	24 02	RCL 2
06	24 03	RCL 3
07	51	+
08	41	-
09	23 04	STO 4
10	74	R/S
11	24 01	RCL 1
12	24 04	RCL 4
13	14 04	f SIN
14	71	\div
15	23 04	STO 4
16	24 02	RCL 2
17	14 04	f SIN
18	61	x
19	74	R/S
20	24 04	RCL 4
21	24 03	RCL 3
22	14 04	f SIN
23	61	x
24	74	R/S

DISPLAY		KEY ENTRY
LINE	CODE	
25	24 01	RCL 1
26	15 02	$g x^2$
27	02	2
28	71	\div
29	24 02	RCL 2
30	14 04	f SIN
31	61	x
32	24 03	RCL 3
33	14 04	f SIN
34	61	x
35	24 02	RCL 2
36	24 03	RCL 3
37	51	+
38	14 04	f SIN
39	71	\div
40	13 00	GTO 00
41		
42		
43		
44		
45		
46		
47		
48		
49		

REGISTERS
R ₀
R ₁ a
R ₂ B
R ₃ C
R ₄ A, (a/sin A)
R ₅
R ₆
R ₇

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	Store a, B, C	a	STO	1			
		B	STO	2			
		C	STO	3			
3	Solve the triangle		f	PRGM	R/S		A°
			R/S				b°
			R/S				c
			R/S				Area
4	If only the area is needed:	a	STO	1			
		B	STO	2			
		C	STO	3			
			GTO	25	R/S		Area
	* The stack must be maintained						
	at these points.						

Example:

Let $a = 20.96$, $B = 64^\circ 32'$, $C = 35^\circ 06'$.

Solution:

First convert B and C to decimal degrees.

$$A = 80.37^\circ$$

$$b = 19.19$$

$$c = 12.22$$

$$\text{Area} = 115.66$$

HYPERBOLIC FUNCTIONS

This program evaluates the six hyperbolic functions by the following formulas:

$$1. \sinh x = \frac{e^x - e^{-x}}{2}$$

$$2. \cosh x = \frac{e^x + e^{-x}}{2}$$

$$3. \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$4. \operatorname{csch} x = \frac{1}{\sinh x} \quad (x \neq 0)$$

$$5. \operatorname{sech} x = \frac{1}{\cosh x}$$

$$6. \operatorname{coth} x = \frac{1}{\tanh x} \quad (x \neq 0)$$

DISPLAY		KEY ENTRY
LINE	CODE	
00		
01	15 07	$g e^x$
02	31	\uparrow
03	15 22	$g 1/x$
04	41	$-$
05	02	2
06	71	\div
07	13 00	GTO 00
08	15 07	$g e^x$
09	31	\uparrow
10	15 22	$g 1/x$
11	51	$+$
12	13 05	GTO 05
13	15 07	$g e^x$
14	31	\uparrow
15	15 22	$g 1/x$
16	41	$-$
17	31	\uparrow
18	31	\uparrow
19	14 73	$f \text{ LAST}x$
20	02	2
21	61	\times
22	51	$+$
23	71	\div
24	13 00	GTO 00

DISPLAY		KEY ENTRY
LINE	CODE	
25		
26		
27		
28		
29		
30		
31		
32		
33		
34		
35		
36		
37		
38		
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41		
42		
43		
44		
45		
46		
47		
48		
49		

REGISTERS
R ₀
R ₁
R ₂
R ₃
R ₄
R ₅
R ₆
R ₇

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	$\sinh x$	x	f	PRGM	R/S		$\sinh x$
	or						
	$\cosh x$	x	GTO	08	R/S		$\cosh x$
	or						
	$\tanh x$	x	GTO	13	R/S		$\tanh x$
	or						
	$\operatorname{csch} x$	x	f	PRGM	R/S		
			g	$1/x$			$\operatorname{csch} x$
	or						
	$\operatorname{sech} x$	x	GTO	08	R/S		
			g	$1/x$			$\operatorname{sech} x$
	or						
	$\coth x$	x	GTO	13	R/S		
			g	$1/x$			$\coth x$

Examples:

- $\sinh 2.5 = 6.05$
- $\cosh 3.2 = 12.29$
- $\tanh 1.9 = 0.96$
- $\operatorname{csch} 4.6 = 0.02$
- $\operatorname{sech} (-.25) = 0.97$
- $\coth (-2.01) = -1.04$

INVERSE HYPERBOLIC FUNCTIONS

This program evaluates the inverse hyperbolic functions by the following formulas:

1. $\sinh^{-1} x = \ln [x + (x^2 + 1)^{1/2}]$
2. $\cosh^{-1} x = \ln [x + (x^2 - 1)^{1/2}] \quad x \geq 1$
3. $\tanh^{-1} x = \frac{1}{2} \ln \left[\frac{1+x}{1-x} \right] \quad x^2 < 1$
4. $\operatorname{csch}^{-1} x = \sinh^{-1} \left[\frac{1}{x} \right] \quad x \neq 0$
5. $\operatorname{sech}^{-1} x = \cosh^{-1} \left[\frac{1}{x} \right] \quad 0 < x \leq 1$
6. $\operatorname{coth}^{-1} x = \tanh^{-1} \left[\frac{1}{x} \right] \quad x^2 > 1$

DISPLAY		KEY ENTRY
LINE	CODE	
00		
01	31	↑
02	31	↑
03	61	x
04	01	1
05	51	+
06	14 02	f√x
07	51	+
08	14 07	f LN
09	13 00	GTO 00
10	31	↑
11	31	↑
12	61	x
13	01	1
14	41	-
15	14 02	f√x
16	51	+
17	14 07	f LN
18	13 00	GTO 00
19	31	↑
20	31	↑
21	01	1
22	51	+
23	21	x↔y
24	32	CHS

DISPLAY		KEY ENTRY
LINE	CODE	
25	01	1
26	51	+
27	71	÷
28	14 07	f LN
29	02	2
30	71	÷
31	13 00	GTO 00
32		
33		
34		
35		
36		
37		
38		
39		
40		
41		
42		
43		
44		
45		
46		
47		
48		
49		

REGISTERS
R ₀
R ₁
R ₂
R ₃
R ₄
R ₅
R ₆
R ₇

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Key in program						
2	$\sinh^{-1} x$	x	f	PRGM	R/S		$\sinh^{-1} x$
	or						
	$\cosh^{-1} x$	x	GTO	10	R/S		$\cosh^{-1} x$
	or						
	$\tanh^{-1} x$	x	GTO	19	R/S		$\tanh^{-1} x$
	or						
	$\operatorname{csch}^{-1} x$	x	g	$1/x$	f	PRGM	
			R/S				$\operatorname{csch}^{-1} x$
	or						
	$\operatorname{sech}^{-1} x$	x	g	$1/x$	GTO	10	
			R/S				$\operatorname{sech}^{-1} x$
	or						
	$\operatorname{coth}^{-1} x$	x	g	$1/x$	GTO	19	
			R/S				$\operatorname{coth}^{-1} x$

Example:

- $\sinh^{-1} (2.4) = 1.61$
- $\cosh^{-1} (90) = 5.19$
- $\tanh^{-1} (-.65) = -0.78$
- $\operatorname{csch}^{-1} (2) = 0.48$
- $\operatorname{sech}^{-1} (.4) = 1.57$
- $\operatorname{coth}^{-1} (3.4) = 0.30$

7

Photograph courtesy of NASA.

HP-25 Program Form

Title _____ Page _____ of _____

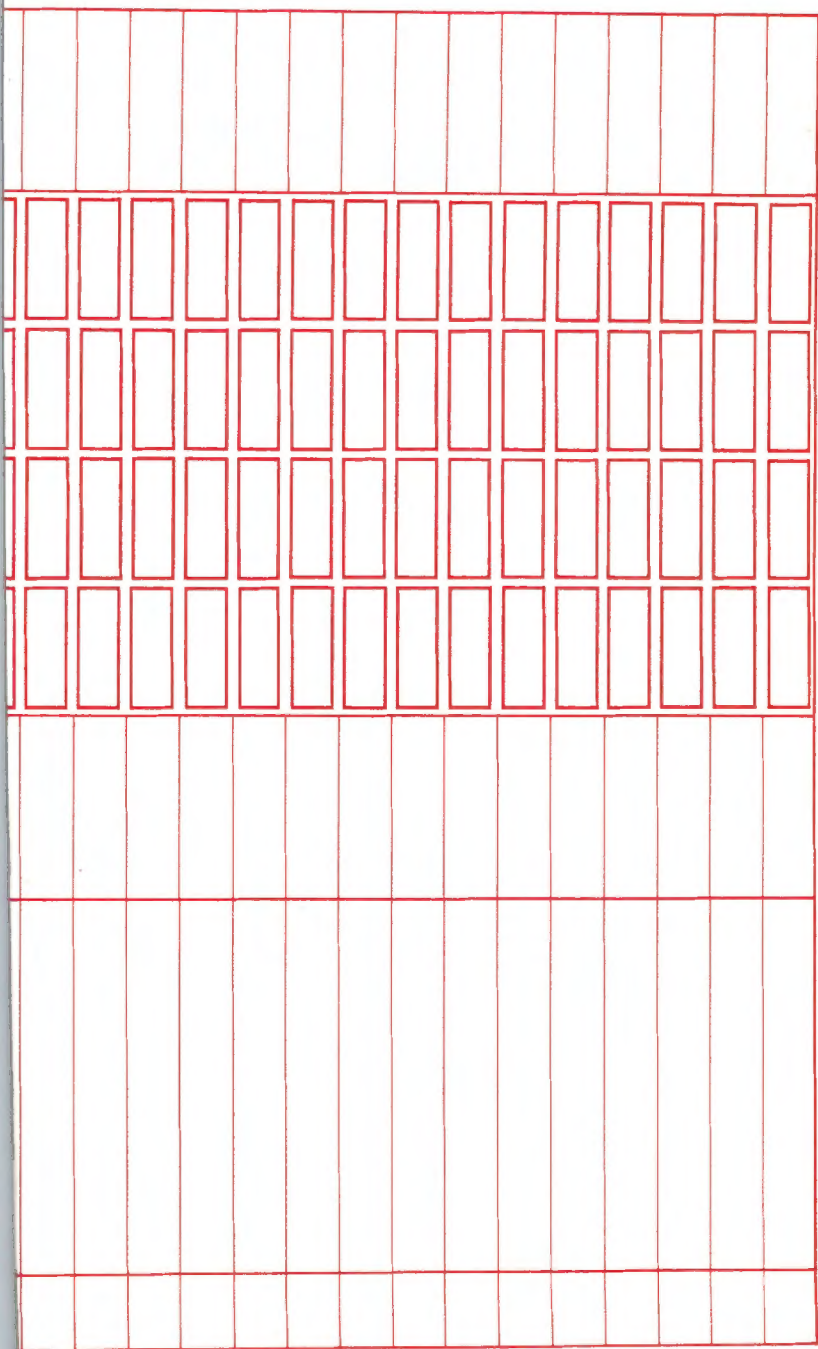
Switch to PRGM mode, press ☐ f ☐ PRGM , then key in the program.

LINE	DISPLAY		KEY ENTRY	X	Y	Z	T	COMMENTS
	CODE							
00								
01								
02								
03								
04								
05								
06								
07								
08								
47								
48								
49								

REGISTERS	
R 0	_____

R 1	_____

R 2	_____



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